Parallel ADMM for robust quadratic optimal resource allocation problems

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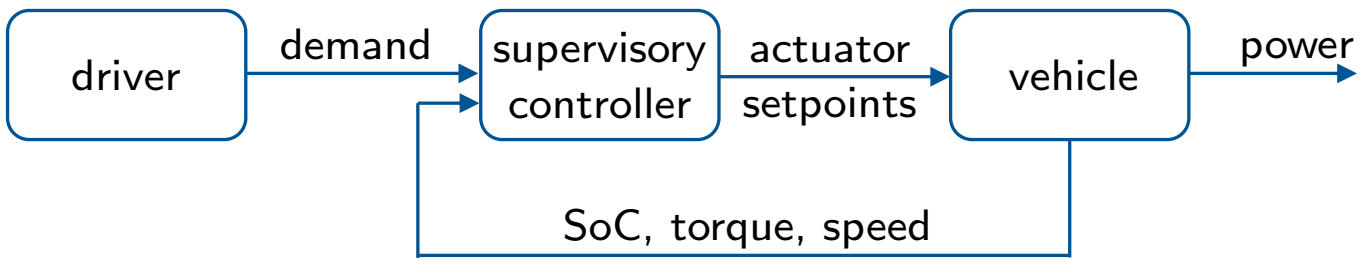
Motivation: PHEV energy management

Optimize power delivered by
i.c. engine and electric motor
while meeting driver demand

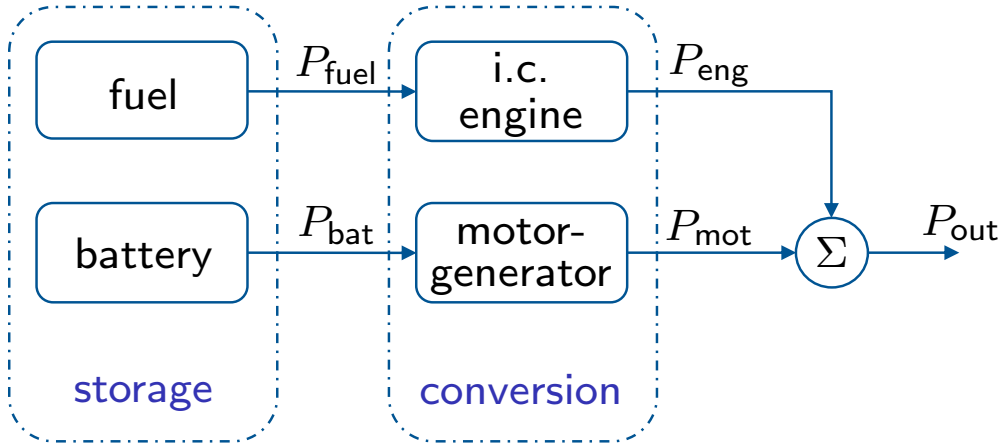


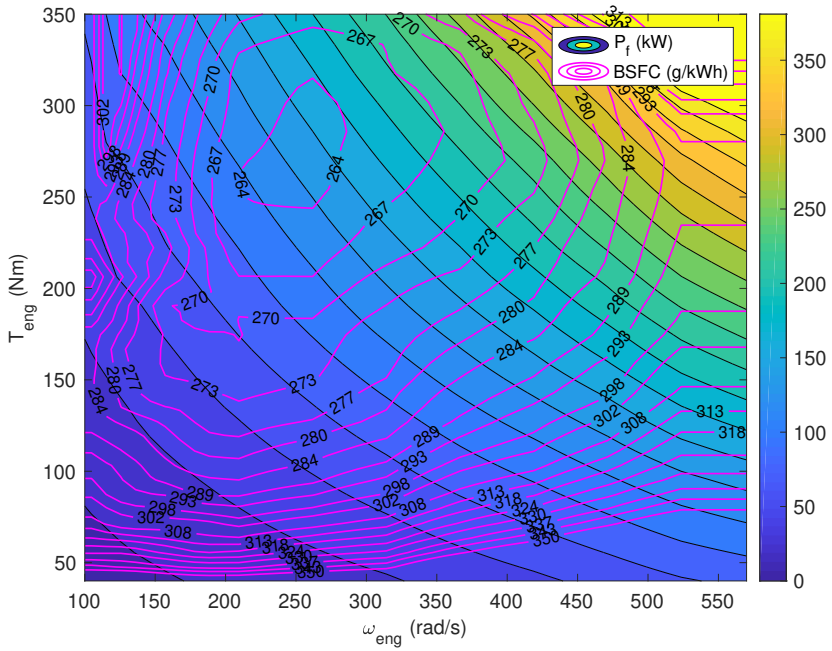
Minimize fuel consumption over a future horizon given

- ⋆ limits on energy capacity (battery SoC, fuel), power flows, torques
- ⋆ predicted demand from planned route, traffic, driver behaviour

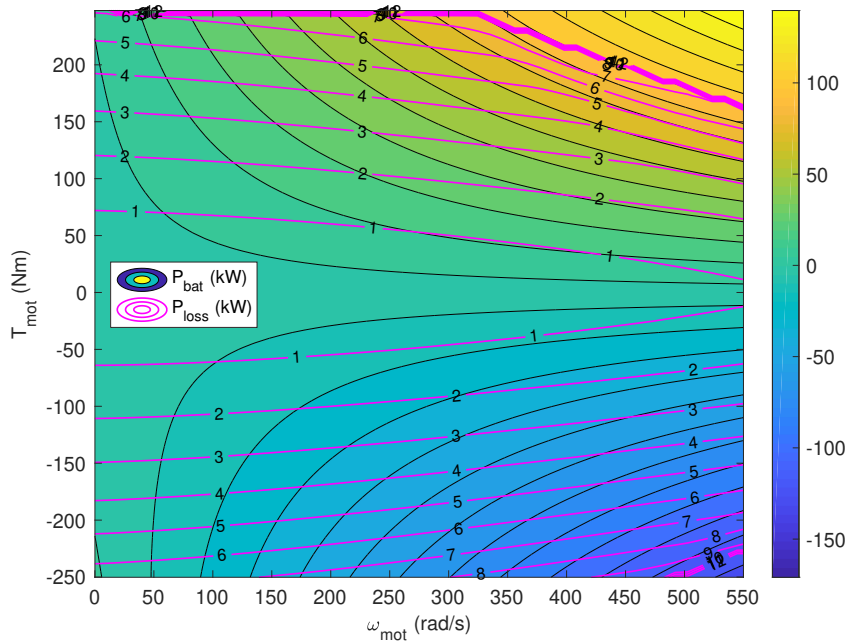
Motivation: PHEV powertrain model

Piecewise quadratic maps fitted to fuel map & electrical loss map





Engine fuel map

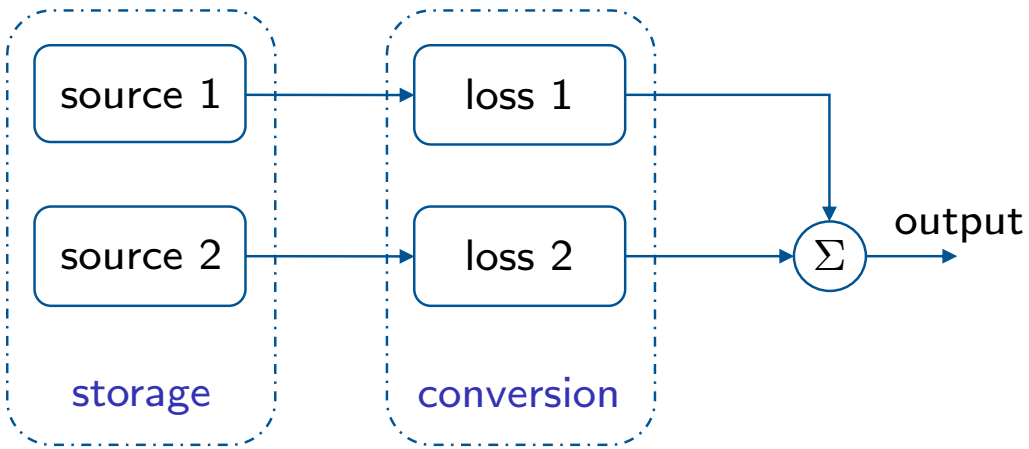


Electrical losses

Motivation: PHEV powertrain model

This paper:

- ▷ Stochastic demand & robust optimization
- ▷ Multiple resources
- ▷ Quadratic losses & quadratic costs
- ▷ Parallel ADMM implementation



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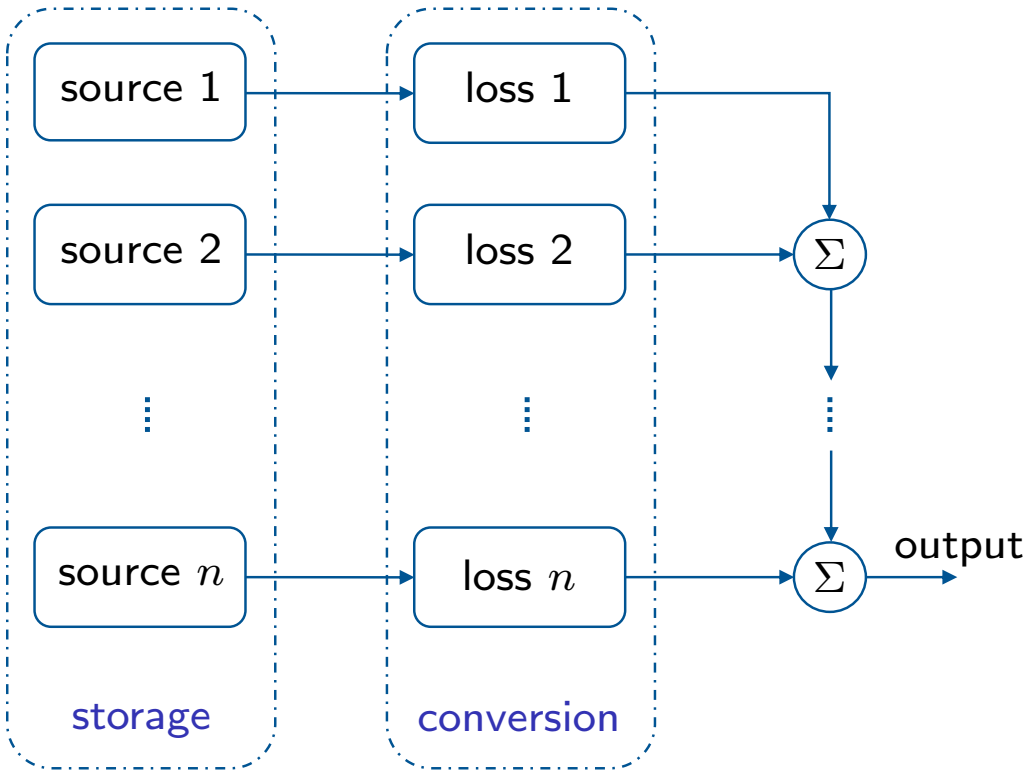
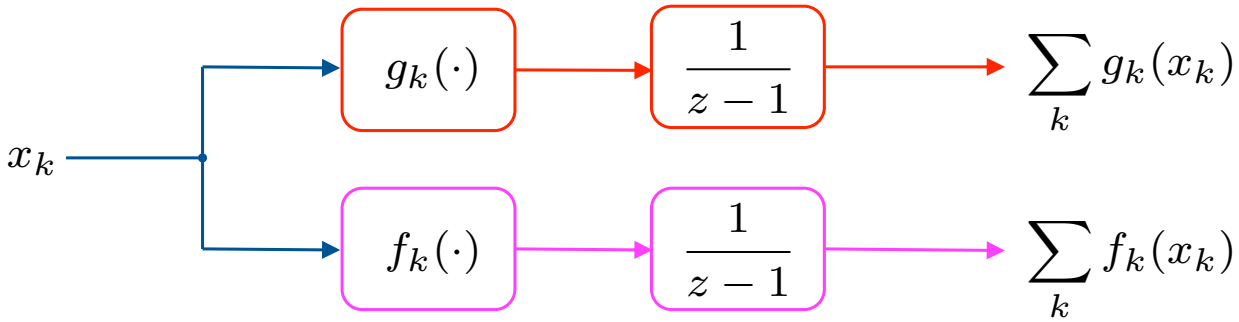


Resource allocation problem definition



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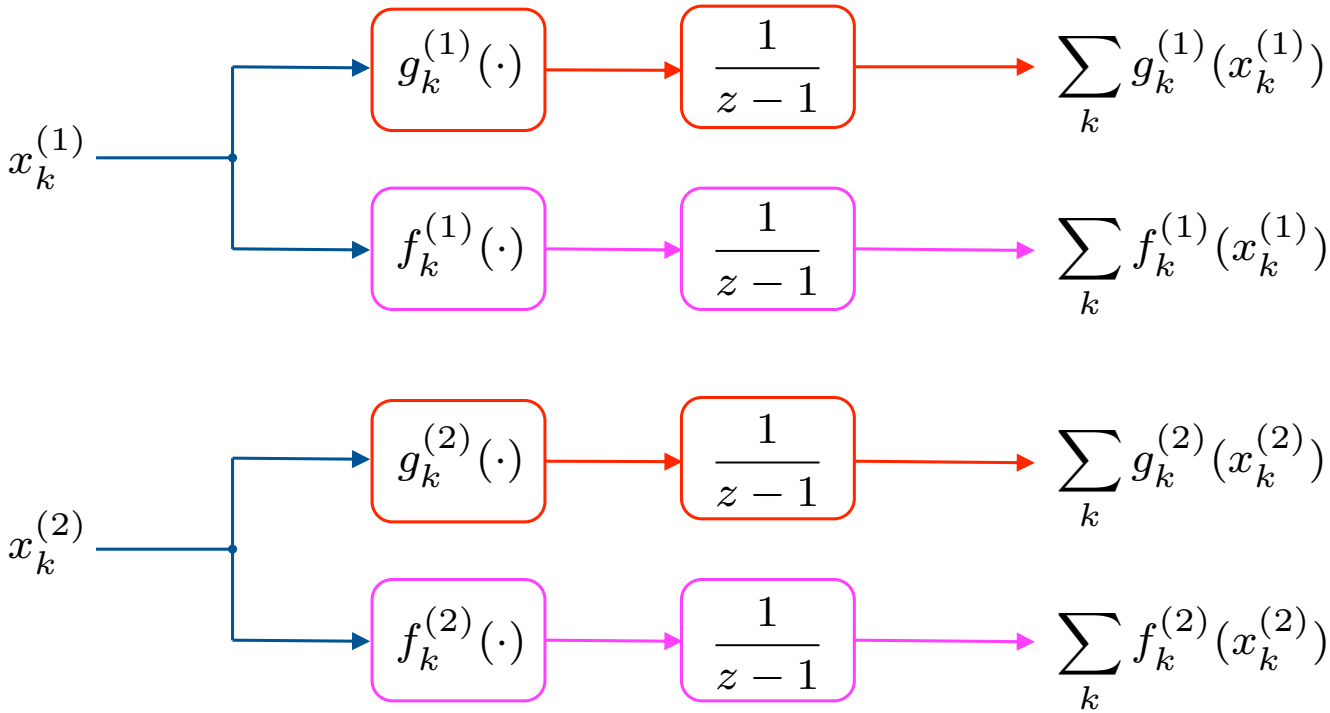

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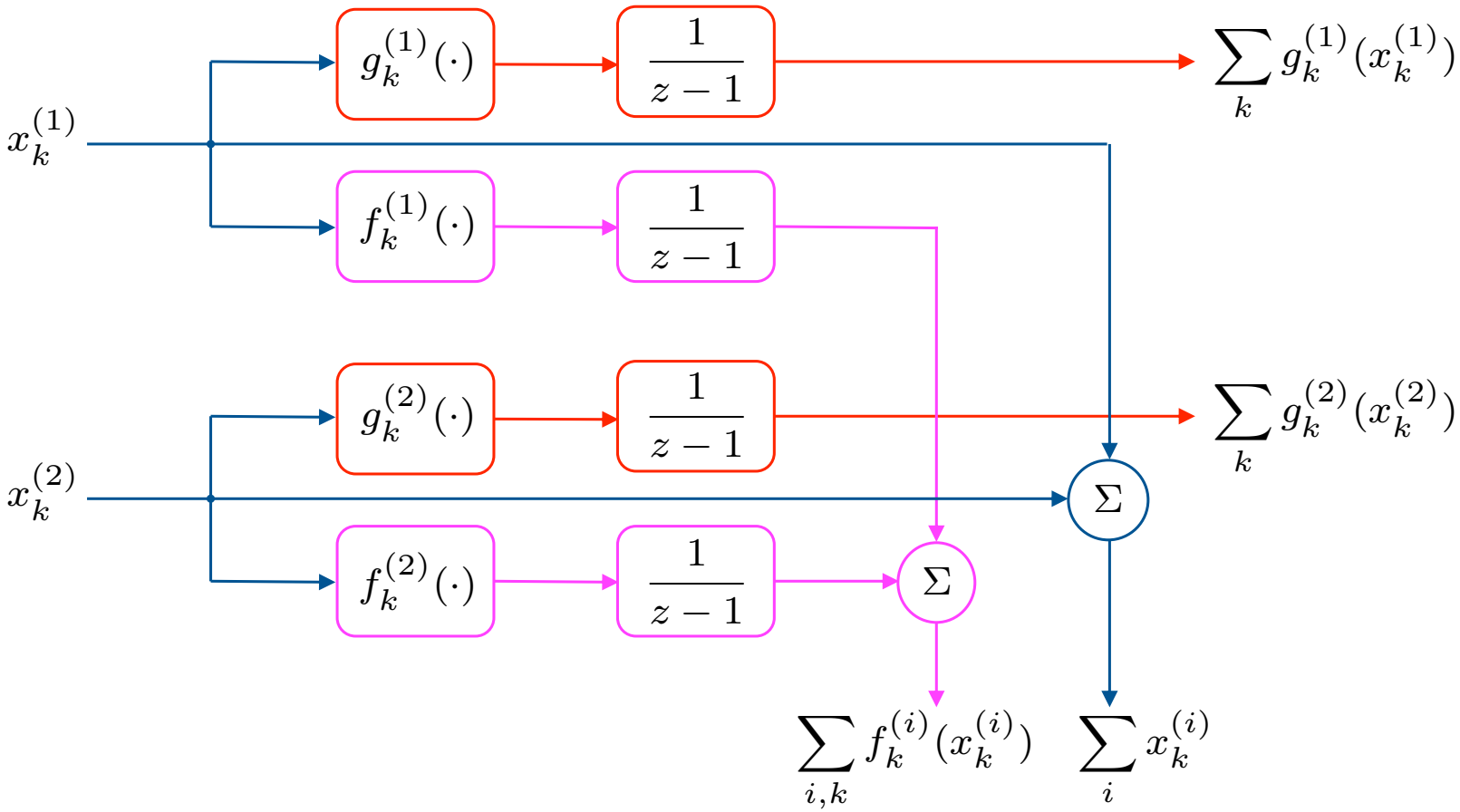

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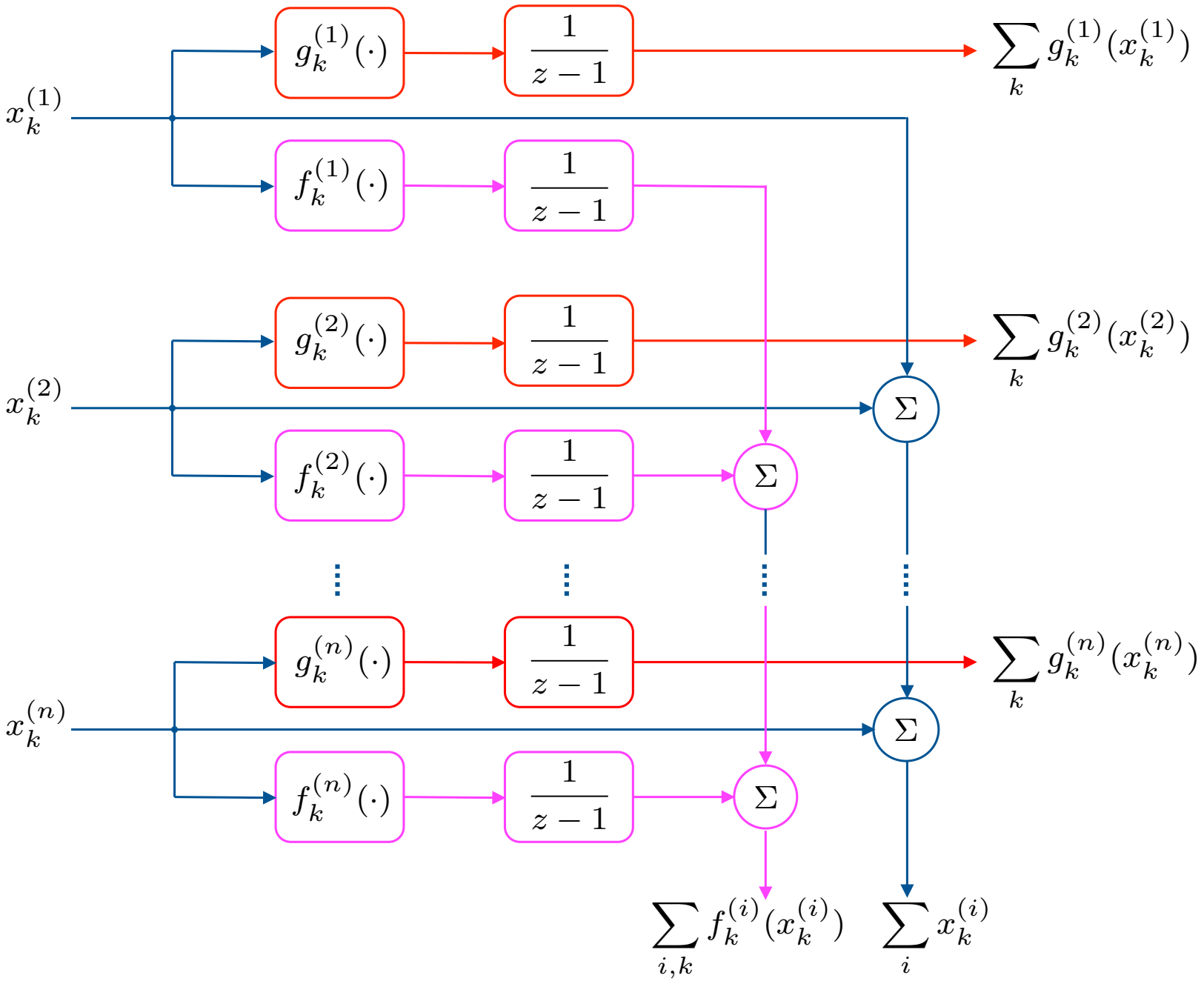

Resource allocation problem definition

If demand sequence: $\{y_1, \ldots, y_n\}$ is known

Optimal allocation for no uncertainty

$$\begin{aligned}
&\underset{x_k^{(i)} \in [\underline{x}_k^{(i)}, \bar{x}_k^{(i)}]}{\text{minimize}} && \sum_{i,k} f_k^{(i)}(x_k^{(i)}) && && \leftarrow \text{total cost} \\
&\text{subject to} && \sum_{i} x_k^{(i)} \geq y_k && \forall k && \leftarrow \text{demand} \\
& && \sum_{k} g_k^{(i)}(x_k^{(i)}) \leq c^{(i)} && \forall i && \leftarrow \text{resource capacity}
\end{aligned}$$

Assumption

$f_k^{(i)}(\cdot)$, $g_k^{(i)}(\cdot)$, are convex and quadratic:

$$f_k^{(i)}(x) = \alpha_{k,2}^{(i)} x^2 + \alpha_{k,1}^{(i)} x + \alpha_{k,0}^{(i)}$$

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Resource allocation problem definition

Unknown demand sequence with samples $y^{(j)} = \{y_1^{(j)}, \ldots, y_n^{(j)}\}$, $j = 1, \ldots, q$

Robust optimal allocation

$$\underset{\substack{x_k^{(i,j)} \in [\underline{x}_k^{(i)}, \bar{x}_k^{(i)}] \\ x_1^{(i)}}}{\text{minimize}} \quad \frac{1}{q} \sum_{i,j,k} f_k^{(i,j)}(x_k^{(i,j)})$$

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Samples $y^{(1)}, \ldots, y^{(q)}$ are i.i.d.

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Algorithm

Equivalent problem:

$$\begin{aligned}
\underset{\substack{x_k^{(i,j)},\ x_1^{(i)} \\ z_k^{(i,j)},\ s_k^{(j)},\ t^{(i,j)}}}{\text{minimize}} \quad & \sum_{i,j,k} \left[\frac{1}{q} f_k^{(i,j)}(x_k^{(i,j)}) + \mathcal{I}_{[\underline{x}_k^{(i)},\, \bar{x}_k^{(i)}]}(x_k^{(i,j)}) \right] \\
& + \sum_{j,k} \mathcal{I}_{\geq 0}(s_k^{(j)}) + \sum_{i,j} \mathcal{I}_{\leq c^{(i)}}(t^{(i,j)}) \\
\text{subject to} \quad & g_k^{(i,j)}(x_k^{(i,j)}) = z_k^{(i,j)} \qquad \forall i, j, k \\
& \sum_i x_k^{(i,j)} - y_k^{(j)} = s_k^{(j)} \qquad \forall j, k \\
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\end{aligned}$$

$$\mathcal{I}_{\mathcal{S}}(x) = \begin{cases} 0 & x \in \mathcal{S} \\ +\infty & x \notin \mathcal{S} \end{cases}$$

Algorithm

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$$\underset{\substack{x_k^{(i,j)},\ x_1^{(i)} \\ z_k^{(i,j)},\ s_k^{(j)},\ t^{(i,j)}}}{\text{minimize}} \quad \sum_{i,j,k} \left[\frac{1}{q} f_k^{(i,j)}(x_k^{(i,j)}) + \mathcal{I}_{[\underline{x}_k^{(i)},\, \bar{x}_k^{(i)}]}(x_k^{(i,j)}) \right]$$

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$$x_1^{(i,j)} = x_1^{(i)} \qquad \forall i, j$$

split capacity constraints into: separable nonlinearities & linear constraints

Algorithm

Augmented Lagrangian:

$$
\begin{aligned}
L = & \sum_{i,j,k} \tfrac{1}{q} f_k^{(i,j)}(x_k^{(i,j)}) \\
& + \sum_{i,j,k} \left[\mathcal{I}_{[\underline{x}_k^{(i)}, \bar{x}_k^{(i)}]}(x_k^{(i,j)}) + \tfrac{\rho_1}{2} \left(z_k^{(i,j)} - g_k^{(i,j)}(x_k^{(i,j)}) + \lambda_k^{(i,j)} \right)^2 \right] \\
& + \sum_{i,j} \left[\mathcal{I}_{\leq c^{(i)}}(t^{(i,j)}) + \tfrac{\rho_2}{2} \left(t^{(i,j)} - \sum_k z_k^{(i,j)} + p^{(i,j)} \right)^2 \right] \\
& + \sum_{j,k} \left[\mathcal{I}_{\geq 0}(s_k^{(j)}) + \tfrac{\rho_3}{2} \left(s_k^{(j)} - \sum_i x_k^{(i,j)} + y_k^{(j)} + \mu_k^{(j)} \right)^2 \right] \\
& + \sum_{i,j} \tfrac{\rho_4}{2} \left(x_1^{(i)} - x_1^{(i,j)} + \nu^{(i,j)} \right)^2
\end{aligned}
$$

- ⋆ $\lambda_k^{(i,j)}$, $\mu_k^{(j)}$, $\nu^{(i,j)}$, $p^{(i,j)}$: Lagrange multipliers
- ⋆ ρ_1, ρ_2, ρ_3: multiplier update step size parameters

Algorithm

ADMM iteration: primal update

$$x_k^{(i,j)} \leftarrow \underset{x_k^{(i,j)}}{\operatorname{argmin}} L = \Pi_{[\underline{x}_k^{(i)}, \bar{x}_k^{(i)}]}\{\text{minimizer of quartic equation in } x_k^{(i,j)}\}$$

$$z_k^{(i,j)} \leftarrow \underset{z_k^{(i,j)}}{\operatorname{argmin}} L =$$

$$g_k^{(i,j)}(x_k^{(i,j)}) - \lambda_k^{(i,j)} + \frac{\rho_2}{\rho_1 + n\rho_2}\left[t^{(i,j)} + p^{(i,j)} - \sum_k \left(g_k^{(i,j)}(x_k^{(i,j)}) - \lambda_k^{(i,j)}\right)\right]$$

$$x_1^{(i)} \leftarrow \underset{x_1^{(i)}}{\operatorname{argmin}} L = \sum_j (x_1^{(i,j)} - \nu^{(i,j)})$$

$$t^{(i,j)} \leftarrow \underset{t^{(i,j)}}{\operatorname{argmin}} L = \Pi_{\leq c^{(i)}}\left\{\sum_k z_k^{(i,j)} - p^{(i,j)}\right\}$$

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can be implemented in parallel

partially parallelizable

Algorithm

ADMM iteration: dual update

$$\mu_k^{(j)} \leftarrow \mu_k^{(j)} + s_k^{(j)} - \sum_i x_k^{(i,j)} + y_k^{(j)}$$

$$\lambda_k^{(i,j)} \leftarrow \lambda_k^{(i,j)} + z_k^{(i,j)} - g_k^{(i,j)}(x_k^{(i,j)})$$

$$\nu^{(i,j)} \leftarrow \nu^{(i,j)} + x_1^{(i)} - x_1^{(i,j)}$$

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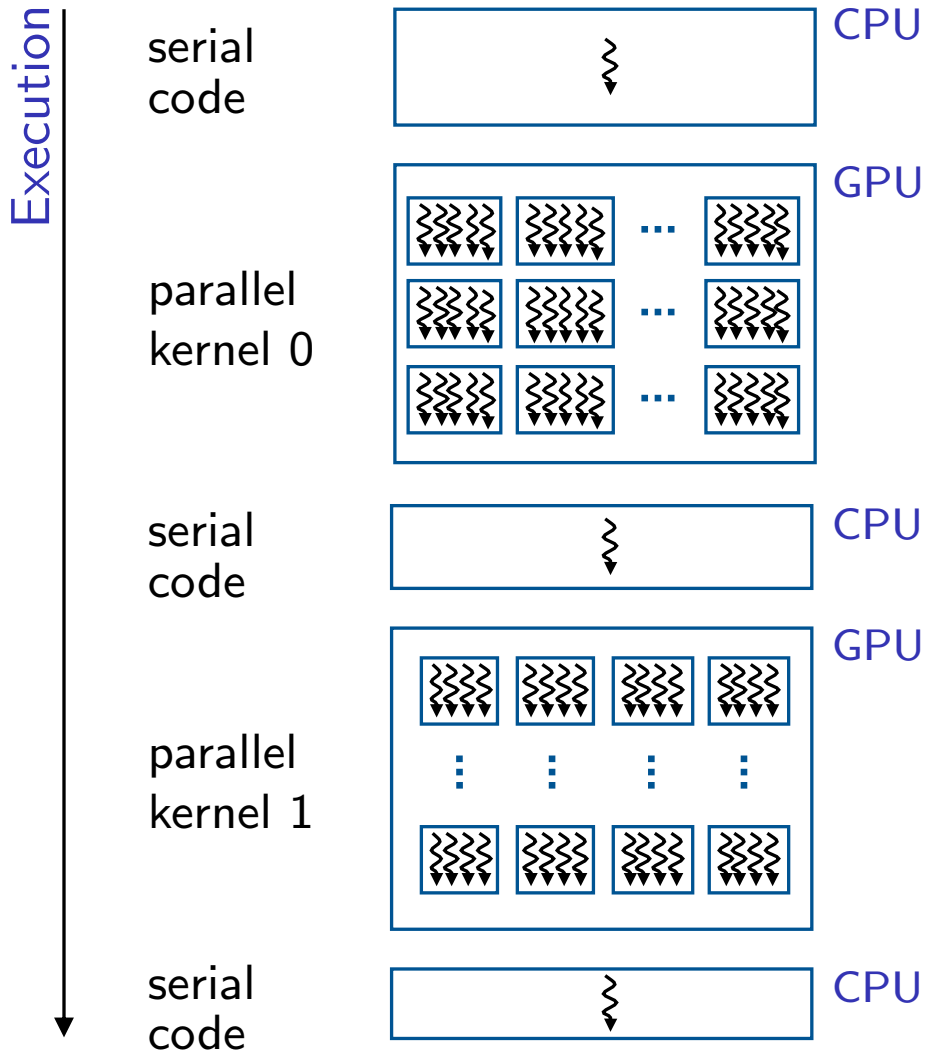
Parallel GPU implementation

CUDA heterogeneous programming model

CUDA kernels run in parallel on the GPU

Threads execute same instructions simultaneously using different data



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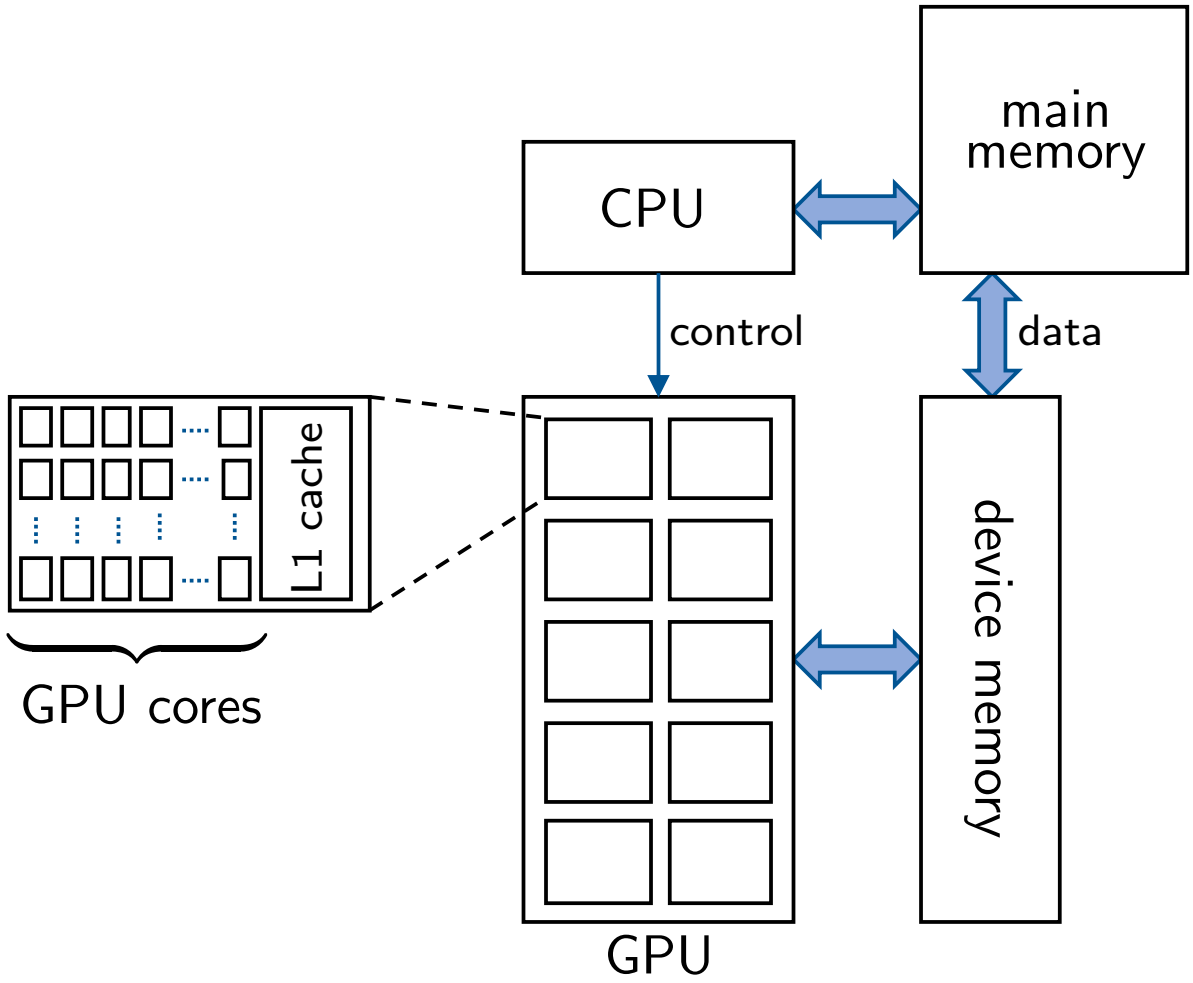


e.g. Nvidia GTX 1060 3GB GPU has 1152 cores and up to 18432 threads

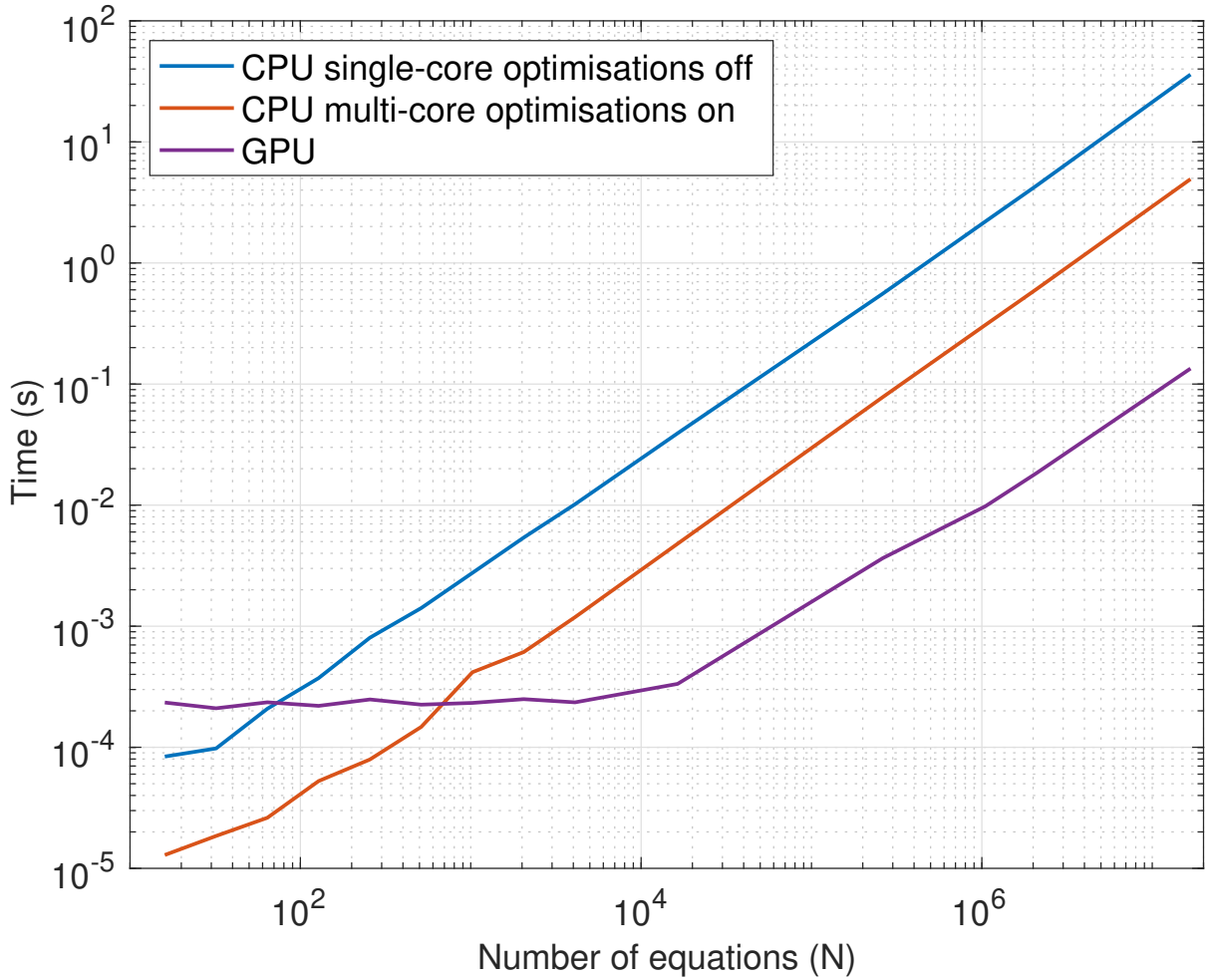
Quartic minimization

- $x^* = \operatorname{argmin}_x Ax^4 + Bx^3 + Cx^2 + Dx$
- Fast algebraic solution based on Vieta's and Cardano's methods:

Input : coefficients A, B, C, D
$b \leftarrow 3B/4A$, $c \leftarrow C/2A$, $d \leftarrow D/4A$;
$Q \leftarrow c/3 - b^2/9$, $R \leftarrow bc/6 - b^3/27 - d/2$, $\Delta \leftarrow Q^3 + R^2$;
if $\Delta > 0$ **then**
  $x^* \leftarrow (R + \sqrt{\Delta})^{1/3} + (R - \sqrt{\Delta})^{1/3} - b/3$;
else if $Q = R = 0$ **then**
  $x^* \leftarrow -b/3$;
else
  $\theta \leftarrow \cos^{-1}(R/|Q|^{3/2})$;
  $x_a \leftarrow 2|Q|^{1/2}\cos(\theta/3) - b/3$;
  $x_b \leftarrow 2|Q|^{1/2}\cos(\theta/3 + 2\pi/3) - b/3$;
  $x_c \leftarrow 2|Q|^{1/2}\cos(\theta/3 + 4\pi/3) - b/3$;
  $(x_1, x_2, x_3) \leftarrow \mathsf{sort}(x_a, x_b, x_c)$;
  $\delta f \leftarrow \frac{1}{4}(x_1^4 - x_3^4) + \frac{b}{3}(x_1^3 - x_3^3) + \frac{c}{2}(x_1^2 - x_3^2) + d(x_1 - x_3)$;
  if $\delta f > 0$ **then**
    $x^* \leftarrow x_3$;
  else
    $x^* \leftarrow x_1$;

Quartic minimization

Computation time for minimization of N quartics with random coefficients
(CPU speed optimizations via compiler flags /Ox and /Od)



Application: PHEV energy management

- $\star$ $y_k^{(j)}$: samples of stochastic driver power demand
- $\star$ $x_k^{(1,j)}$: drive power from i.c. engine
- $\star$ $x_k^{(2,j)}$: drive power from electric motor
- $\star$ Objective: minimize fuel consumption ($f_k^{(2,j)} = 0 \ \forall j,k$)
  constraints: battery capacity & power flows ($g_k^{(1,j)} = 0 \ \forall j,k$)

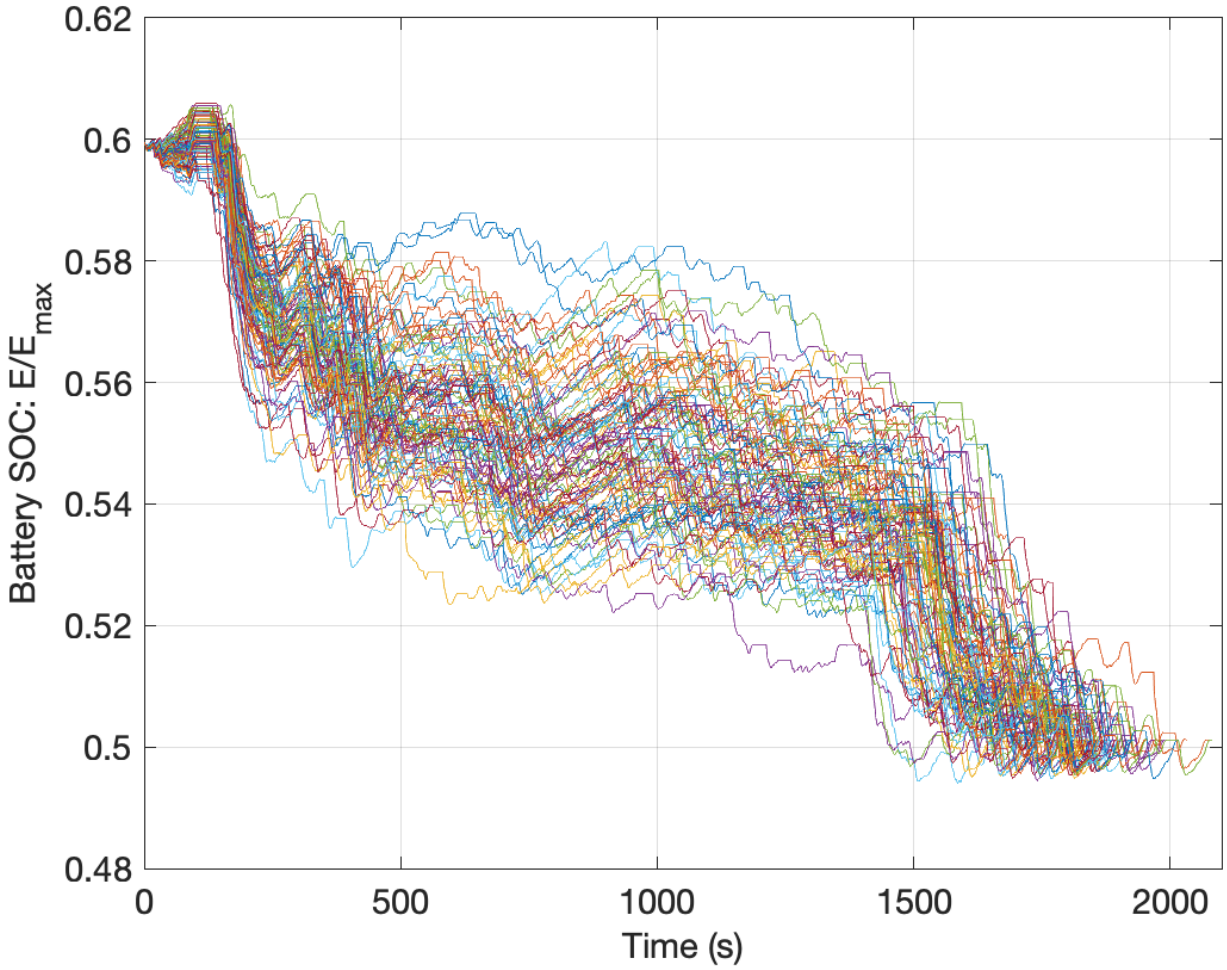
Robust optimization problem

$$\underset{\substack{x_k^{(i,j)} \in [\underline{x}_k^{(i)}, \bar{x}_k^{(i)}] \\ x_1^{(i)}}}{\text{minimize}} \quad \frac{1}{q} \sum_{j=1}^{q} \sum_{k=1}^{n} f_k^{(1,j)}(x_k^{(1,j)})$$

$$\begin{aligned}
\text{subject to} \quad & x_k^{(1,j)} + x_k^{(2,j)} \geq y_k^{(j)}, && j \in \{1,\dots,q\},\ k \in \{1,\dots n\} \\
& \textstyle\sum_k g_k^{(2,j)}(x_k^{(2,j)}) \leq \Delta E, && j \in \{1,\dots,q\} \\
& x_1^{(i,j)} = x_1^{(i)}, && i = 1,2,\ j \in \{1,\dots,q\}
\end{aligned}$$

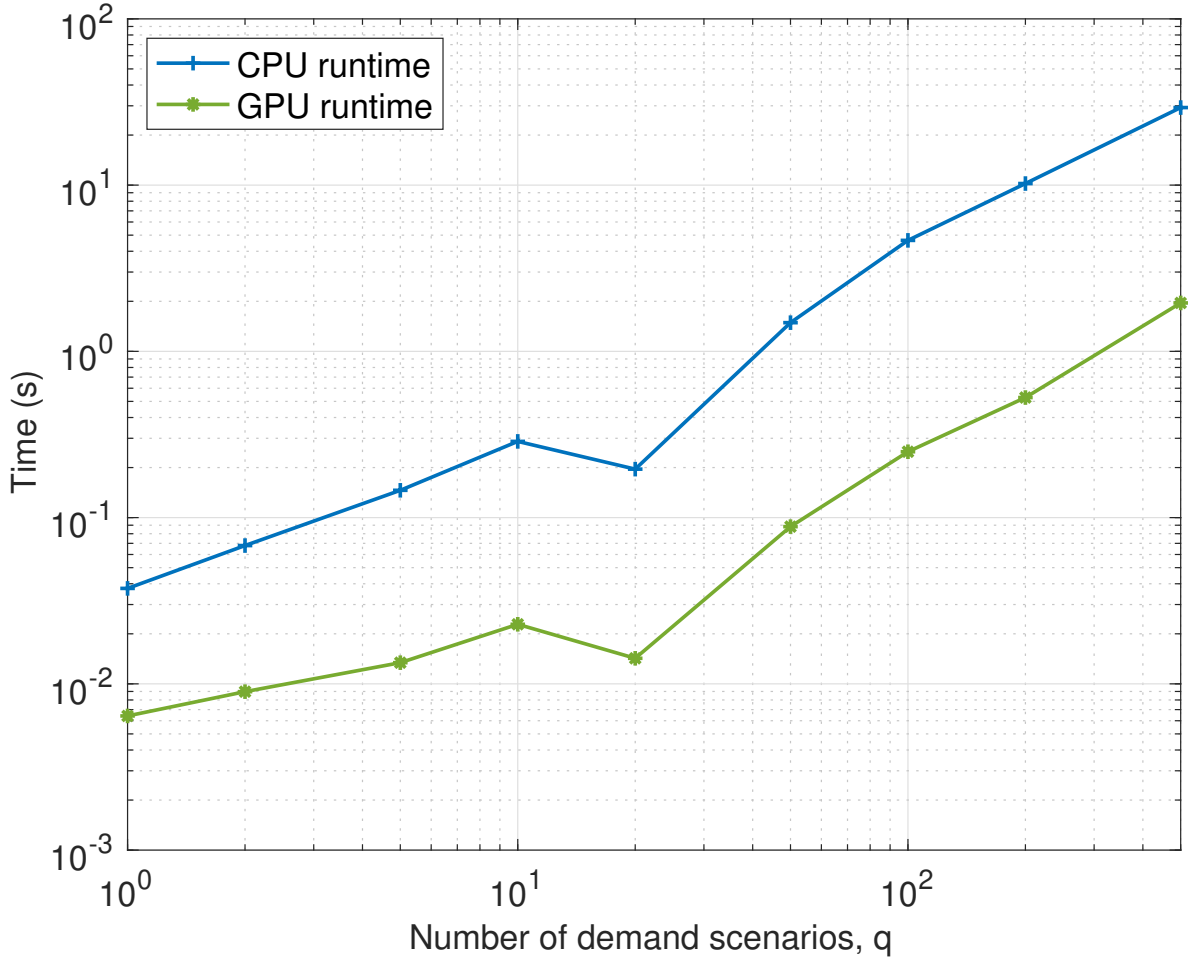
Application: PHEV energy management

Optimal predicted battery state of charge profiles with 100 power demand scenarios generated from random perturbations of FTP-75 drive cycle



Application: PHEV energy management

Average computation times



- ⋆ Parallel implementation is between $10\times$ and $20\times$ faster than serial
- ⋆ 0.25 s for 100 scenarios (0.6 s for 200 scenarios) is acceptable with $T_{\text{samp}} = 1\,\text{s}$

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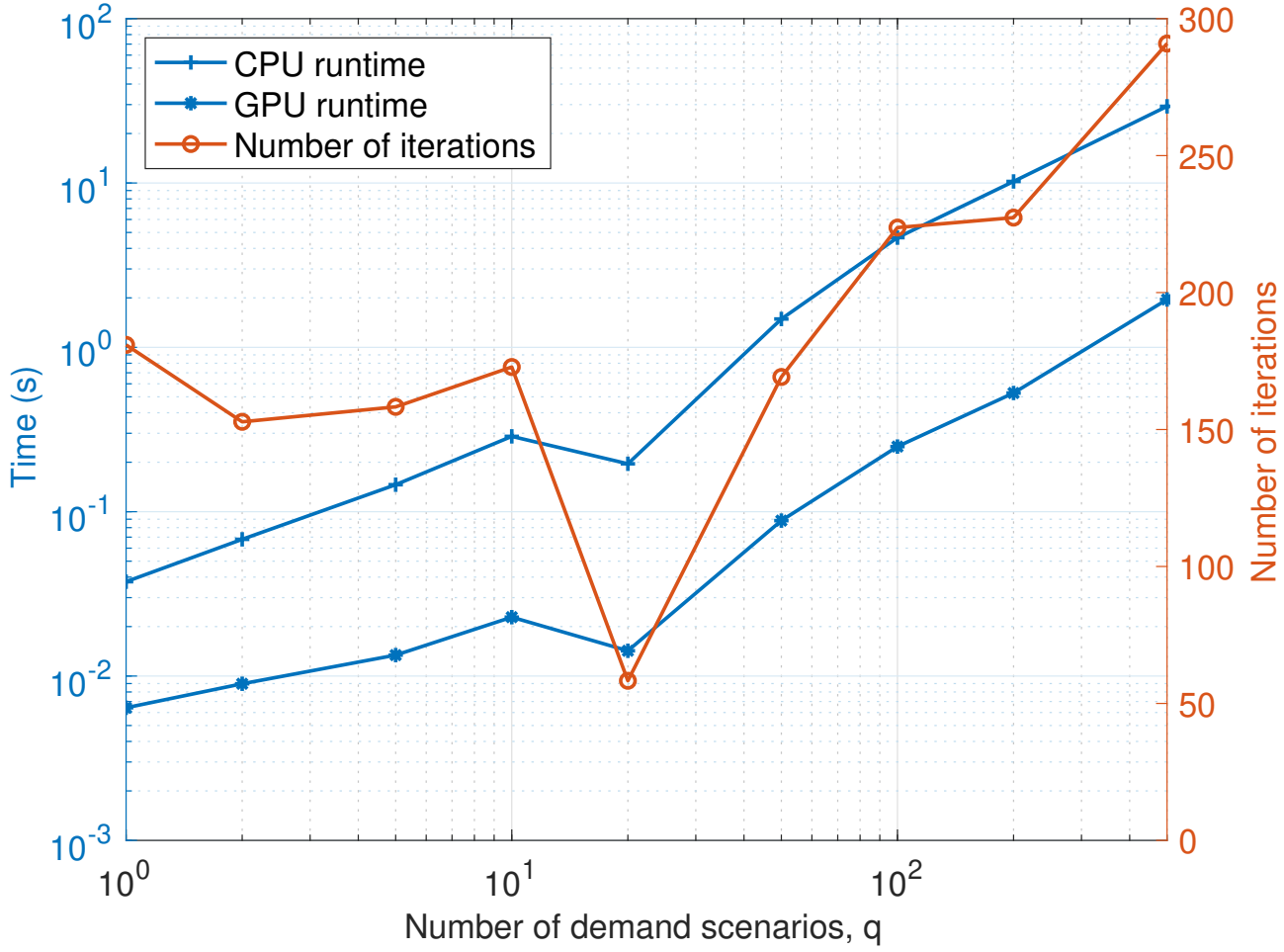

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Conclusions

Contributions

- ADMM algorithm for robust quadratic resource allocation problems
- Parallel implementation on GPU coded in CUDA

Observations

- Choice of operator splitting is important for parallel implementation
- Robust optimal energy management problem is solvable in real time using cheap, non-specialized hardware
- State of the art low-cost parallel processing hardware is evolving fast

Code: https://github.com/qureshizawar/CUDA-quartic-solver

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