Adaptive Model Predictive Control: Robustness and Parameter Estimation

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Robust MPC paradigm:

- MPC requires adequate models of the system, uncertainty, disturbances
- Amount of uncertainty in the model crucially affects performance
- Large effort (time & money) spent on model identification offline
Motivation

Adaptive MPC paradigm:

- Identify model online
- Require: robust constraint satisfaction
  closed loop stability & performance guarantees
  parameter convergence
Motivation

An idea with a long history: e.g. self-tuning control, DMC, GPC . . .
[Clarke, Tuffs, Mohtadi, 1987]

Revisited with new tools:

- Set membership estimation
  [Bai, Cho, Tempo, 1998]

- Robust tube MPC
  [Langsson, Chryssochoos, Rakovic, Mayne, 2004]

- Dual adaptive/predictive control
  [Lee & Lee, 2009]
Motivation

Recent work on MPC with model adaptation

- Focus on online learning & identification:
  - Persistency of Excitation constraints [Marafioti, Bitmead, Hovd, 2014]
  - Kalman filter-based parameter estimation with covariance matrix in cost [Heirung, Ydstie, Foss, 2017]
  - Gaussian process regression, particle filtering [Klenske, Zeilinger, Scholkopf, Hennig, 2016] [Bayard & Schumitzky, 2010]

- Focus on robust constraint satisfaction and performance:
  - Constraints based on prior uncertainty set, online update of cost only [Aswani, Gonzalez, Sastry, Tomlin, 2013]
  - Set-based identification, stable FIR plant model [Tanaskovic, Fagiano, Smith, Morari, 2014]
Motivation

This talk considers how to

- ensure robust constraint satisfaction;
- update constraints & costs online via set-membership & point estimates;
- enforce parameter convergence via persistency of excitation conditions.

Outline:

1. Set membership parameter estimation
2. Polytopic tube robust MPC
3. Parameter convergence and time-varying parameters
Parameter set estimate

Plant model with unknown parameter vector $\theta^*$ and disturbance $w$:

$$x_{k+1} = A(\theta^*)x_k + B(\theta^*)u_k + w_k$$

Assume the model is affine in $\theta^*$ (assumed constant)

$$x_{k+1} = D_k\theta^* + d_k + w_k$$

$$\begin{cases} D_k = D(x_k, u_k) \\ d_k = A_0 x_k + B_0 u_k \end{cases}$$

with stochastic disturbance $w_k \in \mathcal{W}$ a.s., $\mathcal{W}$ known, compact, polytopic

If $x_k, x_{k-1}, u_{k-1}$ are known, then $\theta^*$ must lie in the “unfalsified set”:

$$\Delta_k = \{ \theta : x_k = D_{k-1}\theta + d_{k-1} + w, \ w \in \mathcal{W} \}$$

Hence update the parameter set estimate $\Theta_k$ via

$$\Theta_k = \Theta_{k-1} \cap \Delta_k$$
Parameter set estimate

- If $\Theta_0$ is a compact polytope, then $\Theta_k$ is a compact polytope for all $k > 0$
  But the update $\Theta_k = \Theta_{k-1} \cap \Delta_k$ has potentially unbounded complexity!

- Instead, use fixed complexity sets defined for given $H_\Theta$ by
  \[
  \Theta_k = \{ \theta : H_\Theta \theta \leq h_k \}
  \]
  and update $\Theta_k$ by solving a set of linear programs:
  \[
  [h_k]_i = \max_{w \in \mathcal{W}, \theta \in \Theta_{k-1}} [H_\Theta]_i \theta \quad \text{s.t.} \quad x_k = D_{k-1} \theta + d_{k-1} + w
  \]

Then
  \[
  \Theta_{k-1} \cap \Delta_k \subseteq \Theta_k \subseteq \Theta_{k-1}
  \]
  since
  - $[H_\Theta]_i \theta \leq [h_k]_i$ for all $\theta \in \Delta_k \cap \Theta_{k-1}$  \implies $\Theta_{k-1} \cap \Delta_k \subseteq \Theta_k$
  - $[h_k]_i \leq \max_{\theta \in \Theta_{k-1}} [H_\Theta]_i \theta = [h_{k-1}]_i$  \implies $\Theta_k \subseteq \Theta_{k-1}$
Parameter point estimate

To ensure closed loop $l^2$ stability, we define the MPC cost in terms of a point estimate $\hat{\theta}_k$ of $\theta^*$, computed using a LMS filter.

Given a parameter estimate $\hat{\theta}_k$, let $\hat{x}_{1|k} = A(\hat{\theta}_k)x_k + B(\hat{\theta}_k)u_k$.

Then for a given parameter update gain $\mu > 0$ satisfying

$$1/\mu > \sup_{(x,u) \in \mathcal{Z}} \|D(x,u)\|^2$$

the point estimate $\hat{\theta}_k$ is defined

$$\tilde{\theta}_k = \hat{\theta}_{k-1} + \mu D^\top(x_{k-1}, u_{k-1})(x_k - \hat{x}_{1|k-1})$$

$$\hat{\theta}_k = \Pi_{\Theta_k}(\tilde{\theta}_k)$$

where $\Pi_{\Theta_k}$ is the Euclidean projection onto $\Theta_k$.

Here $\mathcal{Z}$ is the joint state and control constraint set (assumed bounded) and the point estimate update becomes simply a projection onto $\Theta_k$ if $\mu \to 0$. 
The closed loop $l^2$ gain property is based on the following result

**Lemma (Point estimate)**

If $\sup_{k \in \mathbb{N}} \|x_k\| < \infty$ and $\sup_{k \in \mathbb{N}} \|u_k\| < \infty$, then $\theta_k \in \Theta_k$ for all $k$ and

$$\sup_{T \in \mathbb{N}, w_k \in \mathcal{W}, \theta_0 \in \Theta_0} \frac{\sum_{k=0}^{T} \|\tilde{x}_{1|k}\|^2}{\frac{1}{\mu} \|\hat{\theta}_0 - \theta^*\|^2 + \sum_{k=0}^{T} \|w_k\|^2} \leq 1$$

where $\tilde{x}_{1|k} = A(\theta^*)x_k + B(\theta^*)u_k - \hat{x}_{1|k}$ is the 1-step prediction error.
Control Problem

Consider robust regulation of the system

\[ x_{k+1} = A(\theta)x_k + B(\theta)u_k + w_k \]

with \( \theta \in \Theta_k, w_k \in \mathcal{W} \), subject to the state and control constraints

\[ Fx_k + Gu_k \leq 1 = [1 \cdots 1]^\top \]

Assumption (Robust stabilizability)

There exists a set \( \mathcal{X} = \{x : Vx \leq 1\} \) and feedback gain \( K \) such that \( \mathcal{X} \) is \( \lambda \)-contractive for some \( \lambda \in [0, 1) \), i.e.

\[ V\Phi(\theta)x \leq \lambda 1, \quad \text{for all } x \in \mathcal{X}, \theta \in \Theta_0. \]

where \( \Phi(\theta) = A(\theta) + B(\theta)K \).
Control Problem

State and control input sequences predicted at time $k$: $u_{i|k}, x_{i|k}, i = 0, 1, \ldots$ are expressed in terms of decision variables $v = (v_{0|k}, \ldots, v_{N|k})$:

$$u_{i|k} = \begin{cases} K x_{i|k} + v_{i|k} & i = 0, 1, \ldots \\ K x_{i|k} & \end{cases}$$

The regulation cost is defined in terms of point estimate $\hat{\theta}_k$:

$$J_N(x_k, \hat{\theta}_k, v_k) = \sum_{i=0}^{N-1} \left( \|\hat{x}_{i|k}\|_Q^2 + \|\hat{u}_{i|k}\|_R^2 \right) + \|\hat{x}_{N|k}\|_P^2$$

where $\hat{x}_{i|k}, \hat{u}_{i|k}$ are defined by

$$\hat{x}_{i+1|k} = A(\hat{\theta}_k)\hat{x}_{i|k} + B(\hat{\theta}_k)\hat{u}_{i|k}$$

$$\hat{u}_{i|k} = K\hat{x}_{i|k} + v_{i|k}$$

and $P \succeq \Phi^\top(\theta)P\Phi(\theta) + Q + K^\top R K$ for all $\theta \in \Theta_0$
Tube MPC

A sequence of sets (a “tube”) is constructed to bound the predicted state \( x_{i|k} \), with \( i \)th cross section, \( \mathcal{X}_{i|k} \):

\[
\mathcal{X}_{i|k} = \{ x : V x \leq \alpha_{i|k} \}
\]

where \( V \) is determined offline and \( \alpha_{i|k} \) are online decision variables

- For robust satisfaction of \( x_{i|k} \in \mathcal{X}_{i|k} \), we require

  \[
  V\Phi(\theta)x + VB(\theta)v_{i|k} + \bar{w} \leq \alpha_{i+1|k} \quad \text{for all } x \in \mathcal{X}_{i|k}, \ \theta \in \Theta_k
  \]

  where \( [\bar{w}]_i = \max_{w \in \mathcal{W}} [V]_iw \)

- For robust satisfaction of \( Fx_{i|k} + Gu_{i|k} \leq 1 \), we require

  \[
  (F + GK)x + Gv_{i|k} \leq 1 \quad \text{for all } x \in \mathcal{X}_{i|k}
  \]

Condition (A) is bilinear in \( x \) and \( \theta \), but it can be expressed in terms of linear inequalities using a vertex representation of either \( \mathcal{X}_{i|k} \) or \( \Theta_k \).
We generate the vertex representation:

\[ \mathcal{X}_{i|k} = \text{co}\{x_{i|k}^1, \ldots, x_{i|k}^m\} \]

using the property that \( \{x : [V]_r x \leq [\alpha_{i|k}]_r\} \) is a supporting hyperplane of \( \mathcal{X}_{i|k} \) for each \( r \):

Hence each vertex \( x_{i|k}^j \) is given by the intersection of hyperplanes corresponding to a fixed set of rows of \( V \), and

\[ x_{i|k}^j = U^j \alpha_{i|k} \]

for some \( U^j \), determined offline from the vertices of \( \mathcal{X} = \{x : Vx \leq 1\} \)
Tube MPC

In terms of both hyperplane and the vertex descriptions of $\mathcal{X}_{i|k}$, the robust tube constraints become

\[ V\Phi(\theta)U^j\alpha_{i|k} + VB(\theta)v_{i|k} + \bar{w} \leq \alpha_{i+1|k} \text{ for all } \theta \in \Omega_k, \ j = 1, \ldots, m \]

\[ (F + GK)U^j\alpha_{i|k} + Gv_{i|k} \leq 1, \ j = 1, \ldots, m \]

Now condition (B) is linear and (A) can be equivalently written as linear constraints using

**Lemma (Polyhedral set inclusion)**

Let $\mathcal{P}_i = \{x : F_ix \leq f_i\} \subset \mathbb{R}^n$ for $i = 1, 2$. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ iff

\[ \exists \Lambda \geq 0 \text{ such that } \Lambda F_1 = F_2 \text{ and } \Lambda f_1 \leq f_2 \]
Robust MPC online optimization problem

Summary of constraints in the online MPC optimization at time $k$:

$$V x_k \leq \alpha_{0|k}$$

$$\Lambda_{i|k}^j H \Theta = V D(U^j \alpha_{i|k}, KU^j \alpha_{i|k} + v_{i|k})$$

$$\Lambda_{i|k}^j h_k \leq \alpha_{i+1|k} - V d(u^j \alpha_{i|k}, KU^j \alpha_{i|k} + v_{i|k}) - \bar{w}$$

$$\Lambda_{i|k}^j \geq 0$$

$$(F + GK)U^j \alpha_{i|k} + Gv_{i|k} \leq 1$$

$$\Lambda_{N|k}^j H \Theta = V D(U^j \alpha_{N|k}, KU^j \alpha_{N|k})$$

$$\Lambda_{N|k}^j h_k \leq \alpha_{N|k} - V d(u^j \alpha_{N|k}, KU^j \alpha_{N|k}) - \bar{w}$$

$$\Lambda_{N|k}^j \geq 0$$

$$(F + GK)U^j \alpha_{N|k} \leq 1$$

for $i = 0, \ldots, N - 1, j = 1, \ldots, m$

Let $\mathcal{D}(x_k, \Theta_k)$ be the feasible set for the decision variables $v_k, \alpha_k, \Lambda_k$
Robust adaptive MPC algorithm

Offline: Choose $\Theta_0$, $\mathcal{X}$, feedback gain $K$, and compute $P$

Online, at each time $k = 1, 2, \ldots$:

1. Given $x_k$, update the set $(\Theta_k)$ and point $(\hat{\theta}_k)$ parameter estimates

2. Compute the solution $(v_k^*, \alpha_k^*, \Lambda_k^*)$ of the QP

$$\min_{v_k, \alpha_k, \Lambda_k} J(x_k, \hat{\theta}_k, v_k)$$
subject to $(v_k, \alpha_k, \Lambda_k) \in \mathcal{D}(x_k, \Theta_k)$

3. Apply the control law $u_k^* = Kx_k + v_0^*|_k$
Robust adaptive MPC algorithm

**Theorem (Closed loop properties)**

If \( \theta^* \in \Theta_0 \) and \( D(x_0, \Theta_0) \neq \emptyset \), then for all \( k > 0 \):

1. \( \theta^* \in \Theta_k \)
2. \( D(x_k, \Theta_k) \neq \emptyset \)
3. \( Fx_k + Gu_k \leq 1 \)

and the closed loop system is finite-gain \( l^2 \)-stable, i.e. there exist constants \( c_0, c_1, c_2 > 0 \) such that for all \( T \):

\[
\sum_{k=0}^{T} \|x_k\|^2 \leq c_0 \|x_0\|^2 + c_1 \|\hat{\theta}_0 - \theta^*\|^2 + c_2 \sum_{k=0}^{T} \|w_k\|^2
\]
A numerical example

Second-order linear system with

\[(A(\theta), B(\theta)) = (A_0, B_0) + \sum_{i=1}^{3} (A_i, B_i)\theta_i\]

\[A_0 = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.6 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.042 & 0 \\ 0.072 & 0.03 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]

\[A_2 = \begin{bmatrix} 0.015 & 0.019 \\ 0.009 & 0.035 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ -0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.0397 \\ 0.059 \end{bmatrix}.\]

▷ true parameter \(\theta^* = [0.8 \ 0.2 \ -0.5]\top\), initial set \(\Theta_0 = \{\theta : \|\theta\|_\infty \leq 1\}\).

▷ disturbance uniformly distributed on \(\mathcal{W} = \{w \in \mathbb{R}^2 : \|w\|_\infty \leq 0.1\}\), \(w_k\)

▷ state and input constraints: \([x]_2 \geq -0.3\) and \(u_k \leq 1\).
A numerical example: constraint satisfaction

Figure: Realized closed-loop trajectory from initial condition $x_0 = [3 \ 6]^T$ (red line), predicted state tube at time $k = 0$ (tube cross-sections: blue, terminal set: pink)
A numerical example: constraint satisfaction

Figure: Realized closed-loop trajectory from initial condition $x_0 = [3 \ 6]^	op$ (red line), predicted control tube at time $k = 0$ (tube cross-sections: blue)
Persistent excitation

- A regressor $\Psi_k$ is persistently exciting (PE) if

\[
\beta_1 2\mathbb{I} \leq \frac{1}{l} \sum_{k=k_0+1}^{k_0+l} \Psi_k \Psi_k^\top \leq \beta_2 2\mathbb{I}
\]

for some $\beta_1, \beta_2, l > 0$ and all $k_0$ (Narendra, 1987).

- Define the diameter of $\Theta$ as $\text{dia}(\Theta) = \sup_{\theta_1, \theta_2 \in \Theta} \|\theta_1 - \theta_2\|$.

Convergence of set membership parameter estimate

If the noise bound $w \in \mathcal{W}$ is tight and the regressor $D_k$ is persistently exciting, then $\text{dia}(\Theta_k) \to 0$ with probability one [Bai, Cho, Tempo, 1998].

- $\mathcal{W}$ is a tight noise bound if the support of the probability distribution of $w$ is equal to $\mathcal{W}$. 

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Persistent excitation

Regressor: $\Psi_k = D_k^\top = [A_1 x_k + B_1 u_k \cdots A_p x_k + B_p u_k]^\top$

Consider the PE condition evaluated over a window that includes $n$ past time-steps plus current time:

$$\sum_{k=-n}^{k=0} D_k^\top D_k \succeq \beta_1^2 I$$

This is nonconvex in $u_0 = K x_k + v_0 |_k$, but we can linearise to obtain a convex condition. Thus, let $u_0 = u_0^* + \delta u$, so that

$$D_0^\top D_0 \succeq D(x_0, u_0^*)^\top D(x_0, u_0^*) + D^\top (x_0, u_0^*) [B_1 \delta u \cdots B_p \delta u] + [B_1 \delta u \cdots B_p \delta u]^\top D(x_0, u_0^*)$$

Therefore a sufficient condition for $\sum_{k=-n}^{k=0} D_k^\top D_k \succeq \beta_1^2 I$ is an LMI in $\delta u$:

$$\text{LMI}(\delta u) : \sum_{k=-n}^{k=1} D_k^\top D_k + D(x_0, u_0^*)^\top D(x_0, u_0^*)$$

$$+ D(x_0, u_0^*)^\top [B_1 \delta u \cdots B_p \delta u] + [B_1 \delta u \cdots B_p \delta u]^\top D(x_0, u_0^*) \succeq \beta_1^2 I$$
Robust adaptive MPC algorithm with PE constraint

Offline: Choose $\Theta_0$, $\mathcal{X}$, $\beta_1$, feedback gain $K$, and compute $P$

Online, at each time $k = 1, 2, \ldots$

1. Given $x_k$, update the set $(\Theta_k)$ and point $(\hat{\theta}_k)$ parameter estimates

2. Compute the solution $(v_k^*, \alpha_k^*, \Lambda_k^*)$ of the QP

   $$
   \min_{v_k, \alpha_k, \Lambda_k} J(x_k, \hat{\theta}_k, v_k)
   $$

   subject to $(v_k, \alpha_k, \Lambda_k) \in \mathcal{D}(x_k, \Theta_k)$

3. If

   $$
   \sum_{k=-n}^{k=-1} D_k^T D_k + D(x_0, u_0^*)^T D(x_0, u_0^*) \not\preceq \beta_1 I:
   $$

   (a) Re-run the MPC optimization with $v_{0|k} = v_{0|k}^* + \delta u$ and LMI$(\delta u)$ as additional constraints

   (b) If a feasible solution exists, set $v_{0|k}^* \leftarrow v_{0|k}^* + \delta u^*$

4. Apply the control law $u_k^* = Kx_k + v_{0|k}^*$
A numerical example: parameter set

Figure: Parameter set $\Theta_k$ at time steps $k \in \{0, 1, 2; 10, 25, 50; 100, 500, 5000\}$

<table>
<thead>
<tr>
<th>$\Theta$ set</th>
<th>Volume (%)</th>
<th>Cost*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta_0$</td>
<td>100</td>
<td>62.22</td>
</tr>
<tr>
<td>$\Theta_1$</td>
<td>26.1</td>
<td>61.13</td>
</tr>
<tr>
<td>$\Theta_2$</td>
<td>18.3</td>
<td>61.03</td>
</tr>
<tr>
<td>$\Theta_{10}$</td>
<td>12.7</td>
<td>60.96</td>
</tr>
<tr>
<td>$\Theta_{25}$</td>
<td>8.3</td>
<td>60.93</td>
</tr>
<tr>
<td>$\Theta_{50}$</td>
<td>6.3</td>
<td>60.77</td>
</tr>
<tr>
<td>$\Theta_{100}$</td>
<td>3.4</td>
<td>59.45</td>
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<tr>
<td>$\Theta_{500}$</td>
<td>0.7</td>
<td>57.94</td>
</tr>
<tr>
<td>$\Theta_{5000}$</td>
<td>0.0089</td>
<td>53.95</td>
</tr>
<tr>
<td>$\theta^*$</td>
<td>-</td>
<td>52.70</td>
</tr>
</tbody>
</table>

Table: Volume of $\Theta_k$ as $\Theta_k/\Theta_0 \times 100\%$; Cost* with same initial $x_0$ and constraints
Time-varying parameters

Assumption (time-varying parameters)

There exists a constant $r_\theta$ such that the parameter vector $\theta_k^*$ satisfies $\theta_k^* \in \Theta_0$ for all $k$ and $\|\theta_{k+1}^* - \theta_k^*\| \leq r_\theta$

Define the dilation operator:

$$R_i(\Theta) = \{\theta : H_\Theta \theta \leq h + ir_\theta 1\}$$

Then the parameter set update can be expressed

$$\Theta_k = R_1(\Theta_{k-1} \cap \Delta_k) \cap \Theta_0$$

and $\Theta_k$ is replaced in the tube MPC constraints by

$$\Theta_{i|k} = R_i(\Theta_k) \cap \Theta_0$$
Robust adaptive MPC algorithm with time-varying parameters

Parameter estimate bounds and recursive feasibility properties are unchanged:

**Theorem (Closed loop properties)**

\[
\text{If } \theta^* \in \Theta_0 \text{ and } D(x_0, \Theta_0) \neq \emptyset, \text{ then for all } k > 0: \\
\begin{align*}
1 \quad & \theta^* \in \Theta_k \\
2 \quad & D(x_k, \Theta_k) \neq \emptyset \\
3 \quad & Fx_k + Gu_k \leq 1
\end{align*}
\]

But the LMS filter has an additional tracking error, which invalidates the $l^2$ properties, i.e. “certainty equivalence” no longer applies.

However other performance measures can be used in this context, such as the min-max approach of [Lorenzen, Allgöwer, Cannon, 2017]
A numerical example: time-varying parameters

Figure: Parameter set $\Theta_k$ at times $k \in \{0, 100, 200, 300, 400, 500\}$ for the time-varying system with $r_\theta = 0.01$
A numerical example: time-varying parameters

Figure: Parameter set $\Theta_k$ at times $k \in \{0, 5, 25, 70, 120, 500\}$ for the non-time-varying case for comparison
Conclusions & Outlook

Conclusions:
- Adaptive robust MPC with closed loop guarantees is computationally tractable
- Set-membership parameter estimation and LMS point estimates are obvious choices for MPC cost functions and robust constraints
- Nonconvex PE conditions can be relaxed to convex sufficient conditions

Future work
- How to ensure recursive feasibility with PE constraints?
- Are PE conditions better handled by adding terms to the MPC cost (similar to MPC-based dual control)?
- How can we relax the requirement of prior knowledge of a robustly stabilizing local feedback law?

References: