Output Feedback Stochastic MPC with Packet Losses

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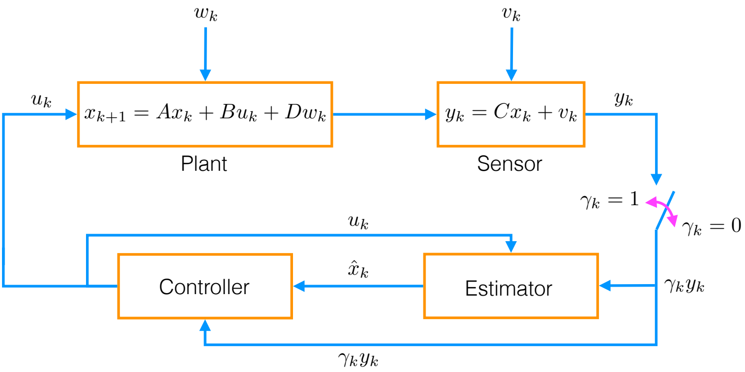
Overview

Output feedback control system



Control problem:

stochastic optimal regulation with linear dynamics, random additive disturbances, random sensor dropout

Plant Model

Linear discrete time system:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + Dw_k \\
y_k &= Cx_k + v_k \\
z_k &= \gamma_k y_k
\end{aligned}
$$

- γ_k, z_k : available to controller
  γ_k : Bernoulli random variable
  $$\mathbb{P}\{\gamma_k = 0\} = 1 - \lambda, \quad \mathbb{P}\{\gamma_k = 1\} = \lambda$$
- w_k, v_k : iid disturbances
  $\mathbb{E}\{w_k\} = 0,\ \mathbb{E}\{v_k\} = 0,\ \mathbb{E}\{w_k w_k^\top\} = \Sigma_w,\ \mathbb{E}\{w_k w_k^\top\} = \Sigma_v$
- λ, Σ_w, Σ_v assumed known
- distributions of w_k, v_k are unknown, possibly unbounded

Control Problem

Discounted performance cost and constraint:

$$\min \sum_{k=0}^{\infty} \beta^k \, \mathbb{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\}$$

$$\text{s.t.} \sum_{k=0}^{\infty} \beta^k \, \mathbb{E}\{\|Hx_k\|^2\} \leq \epsilon$$

$\beta \in (0, 1)$ is a discounting factor

ϵ is a given bound on the constraint function

Control Problem

MPC optimization solved at each sampling time $k = 0, 1, \ldots$:

$$\min \sum_{i=0}^{\infty} \beta^i \, \mathbb{E}_k\{\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\}$$

$$\text{s.t.} \sum_{i=0}^{\infty} \beta^i \, \mathbb{E}_k\{\|Hx_{i|k}\|^2\} \leq \mu_k$$

$x_{\cdot|k}$, $u_{\cdot|k}$: predicted sequences

μ_k is chosen online to ensure recursive feasibility

& closed loop constraint satisfaction

Estimation

State estimate (a priori):

$$\hat{x}_k = \Psi_{k-1}\hat{x}_{k-1} + Bu_{k-1} + \gamma_{k-1}AMy_{k-1}$$

$\Psi_k := A - \gamma_k AMC$

M : static gain chosen so that $\xi_{k+1} = \Psi_k \xi_k$ is mean-square stable

$\Sigma_k := \mathbb{E}_k\big\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^\top\big\}$ evolves according to

$$\Sigma_k = \Psi_{k-1}\Sigma_{k-1}\Psi_{k-1}^\top + \gamma_{k-1}AM\Sigma_v M^\top A^\top + D\Sigma_w D^\top$$

$$\Downarrow$$

$\mathbb{E}_0\{\Sigma_k\}$ is bounded $\forall k > 0$ and $\lim\limits_{k\to\infty} \mathbb{E}_0\{\Sigma_k\}$ exists

Controller parameterization

Predicted control sequence for $i = 0, 1, \ldots$:

$$\begin{aligned} u_{i|k} &= K\hat{x}_{i|k} + c_{i|k} + \gamma_{0|k} L_{i,0|k}(y_{0|k} - C\hat{x}_{0|k}) \\ &\qquad + \gamma_{1|k} L_{i,1|k}(y_{1|k} - C\hat{x}_{1|k}) + \cdots + \gamma_{i|k} L_{i,i|k}(y_{i|k} - C\hat{x}_{i|k}) \\ \hat{x}_{i+1|k} &= A\hat{x}_{i|k} + Bu_{i|k} + \gamma_{i|k} AM(y_{i|k} - C\hat{x}_{i|k}) \end{aligned}$$

- finite parameterization with $c_{i|k} = 0$, $L_{i,j|k} = 0 \ \forall i \geq N$
- affine in the decision variables:

$$\begin{aligned} \theta_k := \big(&\{c_{0|k}, \ldots, c_{N-1|k}\}, \\ &L_{0,0|k},\ \{L_{1,0|k}, L_{1,1|k}\},\ \ldots,\ \{L_{N-1,0|k}, \ldots, L_{N-1,N-1|k}\}\big) \end{aligned}$$

Controller parameterization

Predicted control sequence for $i = 0, 1, \ldots$:

$$\begin{aligned} u_{i|k} &= K\hat{x}_{i|k} + c_{i|k} + \gamma_{0|k} L_{i,0|k}(y_{0|k} - C\hat{x}_{0|k}) \\ &\quad + \gamma_{1|k} L_{i,1|k}(y_{1|k} - C\hat{x}_{1|k}) + \cdots + \gamma_{i|k} L_{i,i|k}(y_{i|k} - C\hat{x}_{i|k}) \\ \hat{x}_{i+1|k} &= A\hat{x}_{i|k} + Bu_{i|k} + \gamma_{i|k} AM(y_{i|k} - C\hat{x}_{i|k}) \end{aligned}$$

Alternative controller parameterizations:

- $u_{i|k} = K\hat{x}_{i|k} + c_{i|k}$ $\rightarrow$ poor performance
- $u_{i|k} = K\hat{x}_{i|k} + c_{i|k} + \gamma_{i|k} KM(y_{i|k} - C\hat{x}_{i|k})$ $\rightarrow$ nonconvexity

Predicted cost and constraint

Define:

vectorized predicted sequences

$$\mathbf{x}_k := \{x_{i|k}\}_{i=0}^{N-1},\ \hat{\mathbf{x}}_k := \{\hat{x}_{i|k}\}_{i=0}^{N-1},\ \mathbf{u}_k := \{u_{i|k}\}_{i=0}^{N-1}$$

2nd moments

$$\mathbf{X}_k := \mathbb{E}_k\left\{ \begin{bmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \hat{\mathbf{x}}_k \end{bmatrix} \begin{bmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \hat{\mathbf{x}}_k \end{bmatrix}^\top \right\}, \quad \mathbf{U}_k := \mathbb{E}_k\{\mathbf{u}_k \mathbf{u}_k^\top\}$$

and decision variables

$$\mathbf{c}_k := \{c_{i|k}\}_{i=0}^{N-1},\ \mathbf{L}_k := \begin{bmatrix} L_{0,0|k} & & & \\ L_{1,0|k} & L_{1,1|k} & & \\ \vdots & \vdots & \ddots & \\ L_{N-1,0|k} & L_{N-1,1|k} & \cdots & L_{N-1,N-1|k} \end{bmatrix}$$

Predicted cost and constraint

Then the cost and constraint function over $i = 0, \ldots, N-1$ are

$$\sum_{i=0}^{N-1} \beta^i \mathbb{E}_k\{\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\} = \mathrm{tr}(\mathbf{Q}_\beta \mathbf{X}_k) + \mathrm{tr}(\mathbf{R}_\beta \mathbf{U}_k)$$

$$\sum_{i=0}^{N-1} \beta^i \mathbb{E}_k\{\|H x_{i|k}\|^2\} = \mathrm{tr}(\mathbf{H}_\beta \mathbf{X}_k)$$

where

$\mathbf{X}_k$ and $\mathbf{U}_k$ are quadratic in $\mathbf{c}_k$ and $\mathbf{L}_k$

Terminal matrix

Define $P_k := \sum_{i=N}^{\infty} \beta^i X_{i|k}$

where $X_{i|k}$ for $i \geq N$ evolves according to

$$X_{i+1|k} = \mathbb{E}\{\tilde{\Psi}(\gamma) X_{i|k} \tilde{\Psi}^\top(\gamma)\} + \mathbb{E}\{\tilde{D}(\gamma) \left[\begin{smallmatrix}\Sigma_v & \\ & \Sigma_w\end{smallmatrix}\right] \tilde{D}^\top(\gamma)\}$$

with

$$\tilde{\Psi}(\gamma) = \begin{bmatrix} A - \gamma AMC & 0 \\ \gamma AMC & A + BK \end{bmatrix}, \quad \tilde{D}(\gamma) = \begin{bmatrix} -\gamma AM & D \\ \gamma AM & 0 \end{bmatrix}$$

$$\Downarrow$$

P_k is the solution of the stochastic Lyapunov equation:

$$P_k = \beta \mathbb{E}\{\tilde{\Psi}(\gamma) P_k \tilde{\Psi}^\top(\gamma)\} + \beta^N X_{N|k} + \frac{\beta^{N+1}}{1-\beta} \mathbb{E}\{\tilde{D}(\gamma) \begin{bmatrix}\Sigma_v & \\ & \Sigma_w\end{bmatrix} \tilde{D}^\top(\gamma)\}$$

Predicted cost and constraint

Then the cost and constraint function over $i = N, N+1, \dots$ are

$$\sum_{i=N}^{\infty} \beta^i \mathbb{E}_k \{ \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \} = \operatorname{tr}\left(\begin{bmatrix} Q & Q \\ Q & Q + K^\top R K \end{bmatrix} P_k \right)$$

$$\sum_{i=N}^{\infty} \beta^i \mathbb{E}_k \{ \|H x_{i|k}\|^2 \} = \operatorname{tr}\left((\mathbf{1}_{2\times 2} \otimes H^\top H) P_k \right)$$

Online MPC optimization

Optimisation solved at time k assuming no knowledge of γ_k:

$$(\mathbf{c}_k^*, \mathbf{L}_k^*, P_k^*) :=$$

$$\arg \min_{\mathbf{c}_k, \mathbf{L}_k, P_k} \mathrm{tr}(\mathbf{Q}_\beta \mathbf{X}_k) + \mathrm{tr}(\mathbf{R}_\beta \mathbf{U}_k) + \mathrm{tr}\left(\begin{bmatrix} Q & Q \\ Q & Q+K^\top RK \end{bmatrix} P_k \right)$$

$$\text{s.t.} \quad \mathrm{tr}(\mathbf{H}_\beta \mathbf{X}_k) + \mathrm{tr}\left[(\mathbf{1}_{2\times 2} \otimes H^\top H) P_k \right] \leq \mu_k,$$

$$P_k \succeq \beta \mathbb{E}\left\{ \tilde{\Psi}(\gamma) P_k \tilde{\Psi}^\top(\gamma) \right\} + \beta^N X_{N|k} + \frac{\beta^{N+1}}{1-\beta} \mathbb{E}\left\{ \tilde{D}(\gamma) \begin{bmatrix} \Sigma_v & \\ & \Sigma_w \end{bmatrix} \tilde{D}^\top(\gamma) \right\}.$$

Control law implemented after receipt of γ_k and $z_k = \gamma_k y_k$:

$$u_k = K\hat{x}_k + c_{0|k}^* + \gamma_k L_{0,0|k}^* (y_k - C\hat{x}_k)$$

Recursive feasibility

To ensure recursive feasibility of the online MPC optimisation, define the constraint threshold μ_k, $\forall k > 0$ as

$$\mu_k := \begin{cases} \epsilon, & k = 0 \\ \mathrm{tr}(\mathbf{H}_\beta \mathbf{X}_k^\circ) + \mathrm{tr}\Big[(\mathbf{1}_{2\times 2} \otimes H^\top H) P_k^\circ\Big], & k > 0 \end{cases}$$

Here $\mathbf{X}_k^\circ$ and P_k° are computed using $(\mathbf{c}_k^\circ, \mathbf{L}_k^\circ)$

where $(\mathbf{c}_k^\circ, \mathbf{L}_k^\circ)$ are a feasible (possibly suboptimal) solution

Recursive feasibility

Feasible (possibly suboptimal) solution at time k:

$$\mathbf{c}_k^\circ := \begin{bmatrix} c^*_{1|k-1} \\ \vdots \\ c^*_{N-1|k-1} \\ 0 \end{bmatrix} + \begin{bmatrix} L^*_{1,0|k-1} \\ \vdots \\ L^*_{N-1,0|k-1} \\ 0 \end{bmatrix} \gamma_{k-1}(y_{k-1} - C\hat{x}_{k-1})$$

$$\mathbf{L}_k^\circ := \begin{bmatrix} L^*_{1,1|k-1} & & & \\ \vdots & \ddots & & \\ L^*_{N-1,1|k-1} & \cdots & L^*_{N-1,N-1|k-1} & \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

Closed loop constraint satisfaction

Theorem 1

If the online MPC optimisation is feasible at $k = 0$, then it remains feasible for all $k > 0$ and the system under the control law satisfies

$$\sum_{k=0}^{\infty} \beta^k \mathbb{E}\{\|Hx_k\|^2\} \le \epsilon$$

Choice of $(\mathbf{c}^\circ_{k+1}, \mathbf{L}^\circ_{k+1})$ ensures that, at time k, the sequences

$$\big(\{x_{i|k+1}\}_{i=0}^{\infty}, \{u_{i|k+1}\}_{i=0}^{\infty}\big) \text{ and } \big(\{x_{i+1|k}\}_{i=0}^{\infty}, \{u_{i+1|k}\}_{i=0}^{\infty}\big)$$

have identical distributions

Therefore

$$\beta\, \mathbb{E}_k\{\mu_{k+1}\} \le \mu_k - \mathbb{E}_k\{\|Hx_{0|k}\|^2\} = \mu_k - \mathbb{E}_k\{\|Hx_k\|^2\}$$

Closed loop cost bound

Theorem 2

Let J_k denote the optimal value of the online MPC optimisation at time k. Then the closed loop system satisfies

$$\sum_{k=0}^{\infty} \beta^k \mathbb{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\} \leq J_0.$$

Let J_k° be the objective value with the feasible solution $(\mathbf{c}_k^\circ, \mathbf{L}_k^\circ)$, then:

$$\beta \mathbb{E}_k\{J_{k+1}^\circ\} = J_k - \mathbb{E}_k\{\|x_k\|_Q^2 + \|u_k\|_R^2\},$$

and by optimality

$$J_k \leq J_k^\circ \ \forall k$$

Numerical example

Linearized model of a double inverted pendulum with:

- w_k and v_k normally distributed
- observations received with probability $\lambda = 0.6$
- discounting factor $\beta = 0.95$
- constraint threshold $\epsilon = 111$

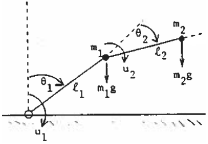


Figure: A double inverted pendulum [1]

Optimal MPC cost at initial time: $J_0 = 2.368 \times 10^4$

[1] E.J. Davison. Benchmark problems for control system design. *Rep. IFAC Theory Committee, 1990*

Numerical Example

Results of 1000 simulations over 500 time steps:

	MPC Controller	LQG Controller
Choice of K	unconstrained LQ-optimal	unconstrained LQ-optimal
Choice of M	steady state Kalman filter gain	time-varying optimal Kalman filter gain
empirical cost value	$4.774 \times 10^3 < J_0$	3.626×10^3
empirical constraint value	$104.7 < \epsilon$	$123.8 > \epsilon$

Conclusions

- Output feedback predicted control policy with affine dependence on future innovation sequences
- Convex formulation
- Recursive feasibility
- Closed loop constraint satisfaction
- Closed loop cost bound

Future work

- Impact of uncertainty in λ on closed loop properties

Questions ?

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