2020 AAS/AIAA Astrodynamics Specialist Conference
August 9-12 2020, South Lake Tahoe

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A multirate variational approach to simulation and optimal control for flexible spacecraft

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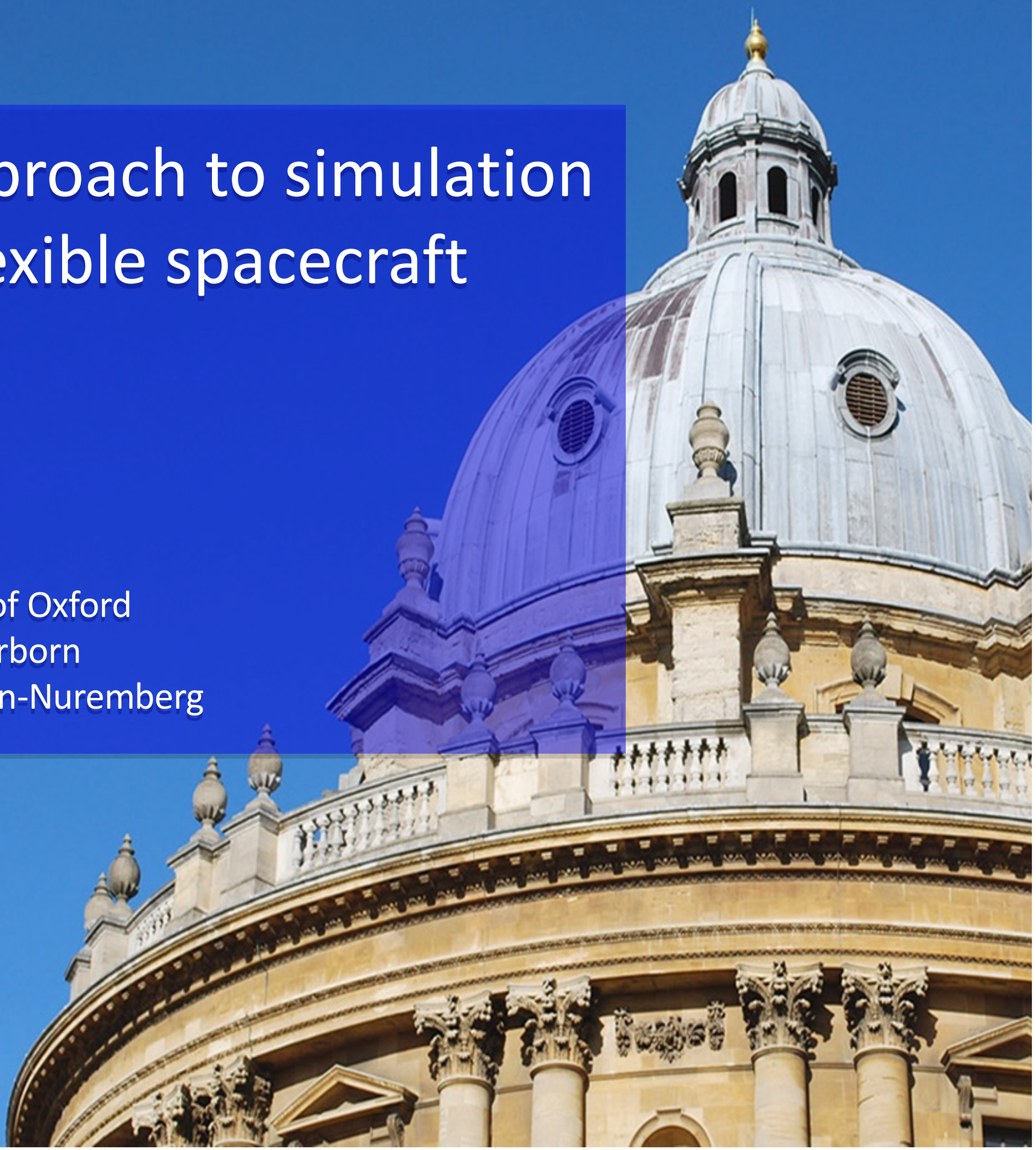


Outline

- Short overview
- Mathematical model
- Multirate DMOC
- Computational results and discussion

Overview

- **Flexible spacecraft**
  - Stringent performance and positioning requirements
  - Efficient operation
  - Lightweight structure designs and induced vibrations- potentially degrading the performance, causing loss of pointing accuracy or even structural damage

-> Need for efficient control method maximizing the system's performance while respecting all hard safety-critical constraints

- **Multirate Discrete Mechanics and Optimal Control (Multirate DMOC)** [1-4] **:**

✓ **High fidelity simulation at a reduced computational cost**

Advantages

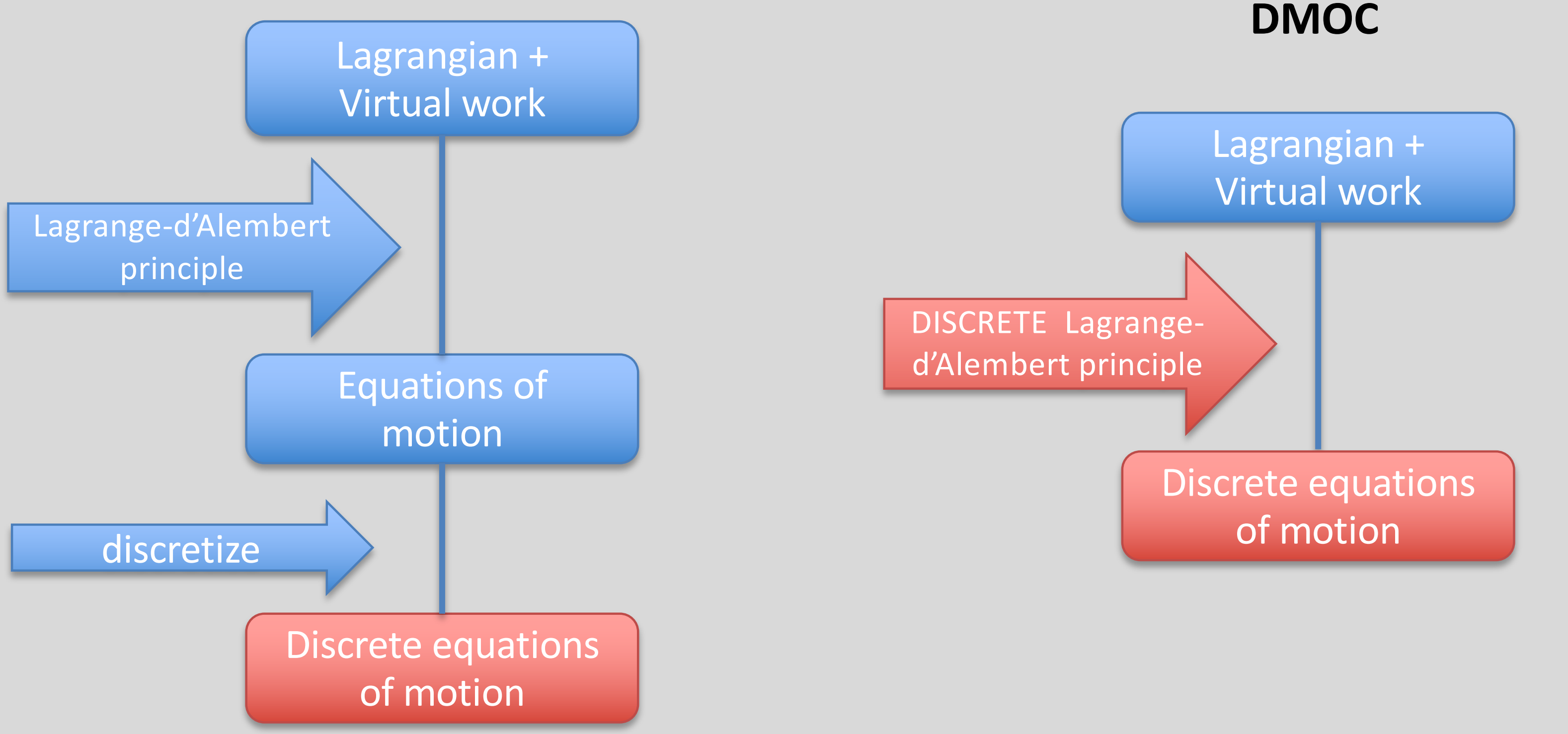
- **Multirate Discrete Mechanics and Optimal Control (Multirate DMOC):**

  High fidelity simulation at a reduced computational cost

  ✓ **Structure-preserving -> *conservation of energy and/or momenta of the system***

Standard direct methods

DMOC



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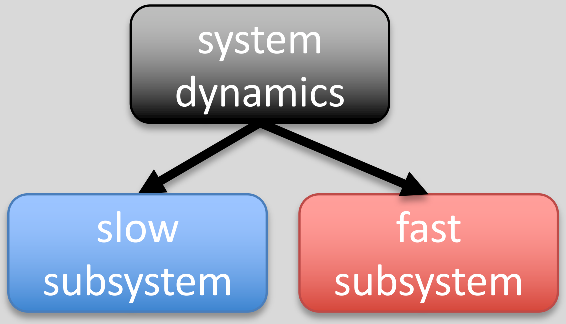
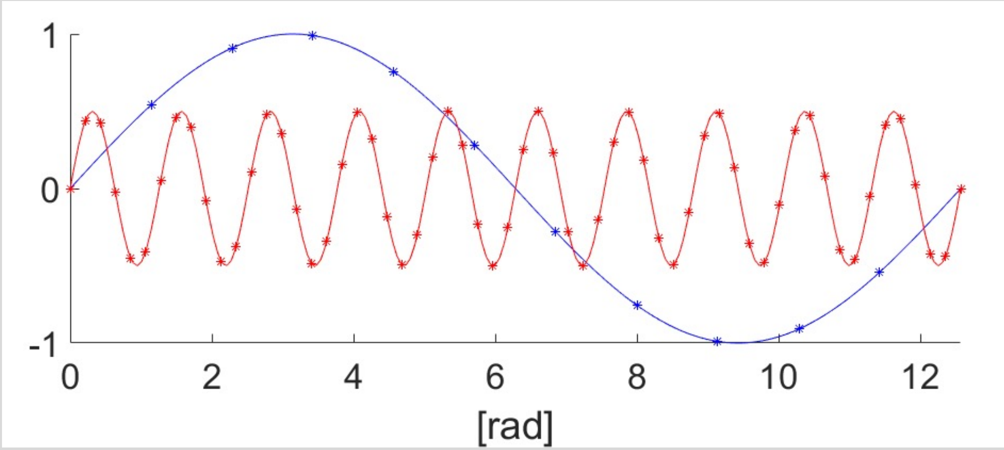
  ✓ **Multirate discretization -> *computational efficiency*** [5]





- **Reduces the number of optimization variables**
- **Reduces the number of equality constraints**

Advantages

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  ✓ Multirate discretization -> *computational efficiency*

  ✓ **Unified control methodology -> *potential improvement in optimality and constraint-handling capabilities****

**For comparison see:* M. Azadi, M. Eghtesad, S. Fazelzadeh, and E. Azadi, “Dynamics and control of a smart flexible satellite moving in an orbit,” *Multibody System Dynamics*, Vol. 35, No. 1, 2015, pp. 1–23.

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  ✓ **No need for decoupling of the equations of motion -> *simple application to a variety of linear and nonlinear models****

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  ✓ **Straightforward selection of slow and fast subsystems -> *capability to tailor the solution to the time scales present in the problem to obtain further reductions in computational cost****

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Mathematical model

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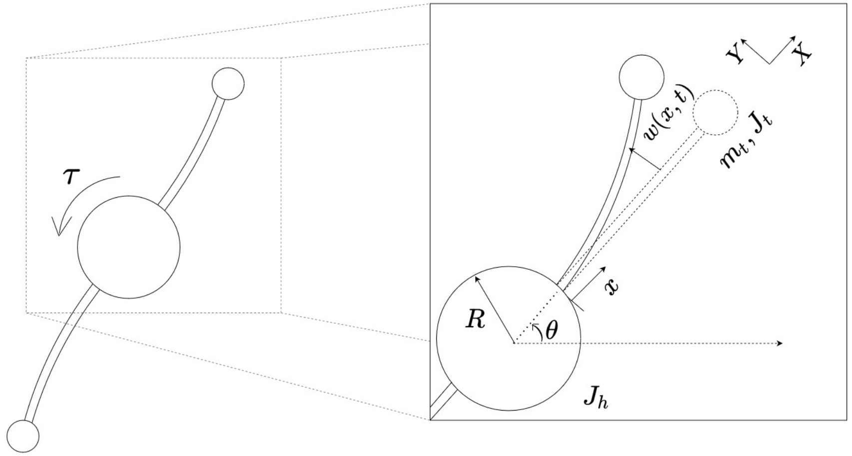
- **General linear model neglecting dissipation effects**
  - **single-axis rest-to-rest rotational maneuver**
  - **control torque applied at the hub**



- **Spatial discretization - The Assumed Modes Method [7]**

$$w(x,t) = \sum_{j=1}^{N} \phi_j(x)\, \eta_j(t), \qquad x \in [0, L]$$

ϕ_j - assumed spatial mode shapes

η_j - time varying modal amplitudes

N - number of modes retained in the approximation

L - length of the beam

- **System description [7]**

$$\underline{\xi} = \begin{bmatrix} \theta \\ \underline{\eta} \end{bmatrix}, \qquad \underline{\eta} = [\eta_1,\, \eta_2,\, \ldots,\, \eta_N]^T \in R^{N\times 1}$$

- **Lagrange - d'Alembert principle**

$$\mathcal{L}(\underline{\xi},\, \dot{\underline{\xi}}) = \frac{1}{2}\, \dot{\underline{\xi}}^T\, \mathbf{M}\, \dot{\underline{\xi}} - \frac{1}{2}\, \underline{\xi}^T\, \mathbf{K}\, \underline{\xi}, \qquad \delta W = \underline{\mathfrak{f}} \cdot \delta\underline{\xi} = \tau\, \delta\theta$$

$$\delta \int_{t_0}^{t_f} \mathcal{L}(\underline{\xi},\, \dot{\underline{\xi}})\, dt + \int_{t_0}^{t_f} (\, \underline{\mathfrak{f}} \cdot \delta\underline{\xi}\,)\, dt = 0 \text{ for all variations } \delta\underline{\xi} \text{ with } \delta\underline{\xi}(t_0) = \delta\underline{\xi}(t_f) = 0$$

$$\mathbf{M}\, \ddot{\underline{\xi}} + \mathbf{K}\, \underline{\xi} = \mathbf{D}\, \tau, \quad \mathbf{M} = \begin{bmatrix} M_{\theta\theta} & M_{\theta\eta}^T \\ M_{\theta\eta} & M_{\eta\eta} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & K_{\eta\eta} \end{bmatrix}, \quad \mathbf{D} = [1, 0, \ldots, 0]^T$$

Transformation to modal coordinates

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$$\underset{\sim}{\xi} = \begin{bmatrix} \theta \\ \underset{\sim}{\eta} \end{bmatrix}$$

$$\mathcal{L}(\underset{\sim}{\xi},\, \dot{\underset{\sim}{\xi}}) = \frac{1}{2}\,\dot{\underset{\sim}{\xi}}^T\,\mathbf{M}\,\dot{\underset{\sim}{\xi}} - \frac{1}{2}\,\underset{\sim}{\xi}^T\,\mathbf{K}\,\underset{\sim}{\xi}$$

$$\underset{\sim}{f} = \mathbf{D}\tau$$

$$\mathbf{M}\,\ddot{\underset{\sim}{\xi}} + \mathbf{K}\,\underset{\sim}{\xi} = \mathbf{D}\,\tau$$

transformation to modal coordinates [7]

$$\underset{\sim}{\xi} = \mathbf{E}\,\underset{\sim}{q}$$

$$\mathcal{L} = \frac{1}{2}(\,\dot{\underset{\sim}{q}}^T\dot{\underset{\sim}{q}} - \underset{\sim}{q}^T\boldsymbol{\Lambda}\,\underset{\sim}{q}\,)$$

$$\underset{\sim}{f} = \mathbf{E}^T\mathbf{D}\,\tau$$

$$\ddot{\underset{\sim}{q}} + \boldsymbol{\Lambda}\,\underset{\sim}{q} = \mathbf{E}^T\mathbf{D}\,\tau$$

$$\mathbf{E}^T\mathbf{M}\mathbf{E} = \mathbf{I}, \quad \mathbf{E}^T\mathbf{K}\mathbf{E} = \boldsymbol{\Lambda}, \qquad \boldsymbol{\Lambda} = \begin{bmatrix} \omega_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{N+1}^2 \end{bmatrix}$$

$$w_1 \le w_2 \le \ldots \le w_{N+1},$$

Table 1: Structural parameters used for the simulations

Hub radius	R	1.0	ft
Hub rotary inertia	J_h	8.0	slug-ft^2
Tip mass	m_t	0.156941	slug
Tip mass rotary inertia	J_t	0.0018	slug-ft^2
Beam length	L	4.0	ft
Beam height	h	6.0	in.
Beam thickness	t	0.125	in.
Beam linear density	ρA	0.0271875	slug/ft
Beam elastic modulus	E	0.1584×10^{10}	lb/ft^2

Table 2: Natural frequencies

w_1	0	rad/s
w_2	6.454	rad/s
w_3	52.41	rad/s
w_4	1.607×10^2	rad/s
w_5	3.381×10^2	rad/s
w_6	5.78×10^2	rad/s

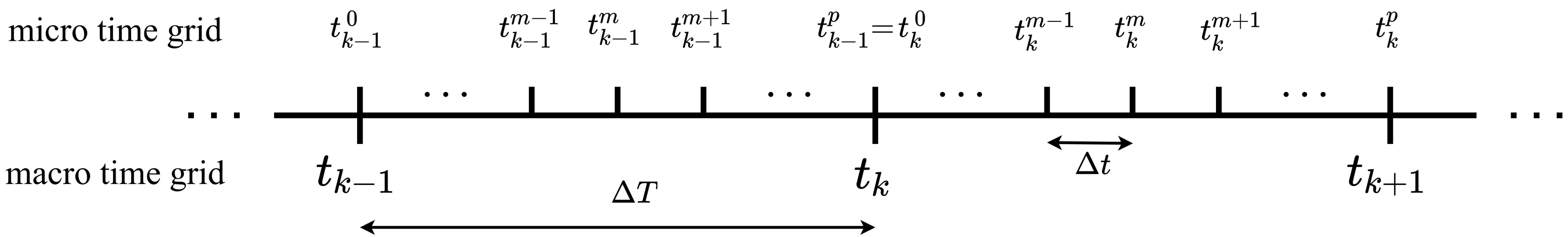
$$N = 5$$

Multirate formulation

$$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} \quad \begin{aligned} &\left.\begin{matrix} q_1 \\ q_2 \\ q_3 \end{matrix}\right\} \underline{q}_s \longrightarrow \begin{aligned} w_1 &= 0 \text{ rad/s} \\ w_2 &= 6.454 \text{ rad/s} \\ w_3 &= 52.41 \text{ rad/s} \end{aligned} \\ &\left.\begin{matrix} q_4 \\ q_5 \\ q_6 \end{matrix}\right\} \underline{q}_f \longrightarrow \begin{aligned} w_4 &= 1.607 \times 10^2 \text{ rad/s} \\ w_5 &= 3.381 \times 10^2 \text{ rad/s} \\ w_6 &= 5.78 \times 10^2 \text{ rad/s} \end{aligned} \end{aligned}$$

$$\mathcal{L} = \frac{1}{2}\Big((\dot{\underline{q}}^s)^T \dot{\underline{q}}^s - (\underline{q}^s)^T \mathbf{\Lambda_s}\, \underline{q}^s \Big) + \frac{1}{2}\Big((\dot{\underline{q}}^f)^T \dot{\underline{q}}^f - (\underline{q}^f)^T \mathbf{\Lambda_f}\, \underline{q}^f \Big)$$

where $\mathbf{\Lambda_s} = \text{diag}(\omega_1^2, \ldots, \omega_3^2)$, $\mathbf{\Lambda_f} = \text{diag}(\omega_4^2, \ldots, \omega_6^2)$



$$p = \frac{\text{macro time step}}{\text{micro time step}} = \frac{\Delta T}{\Delta t}$$

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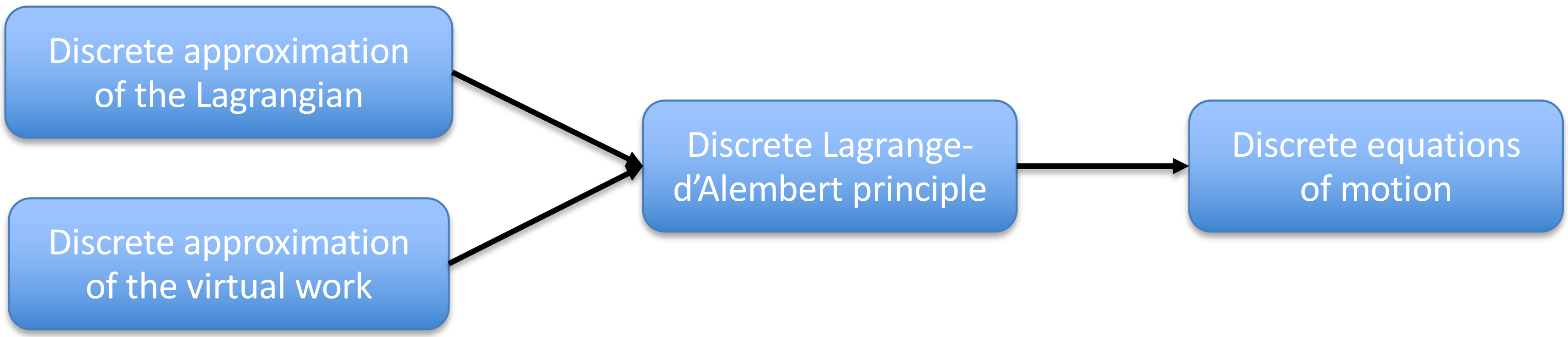


Forward simulation without control

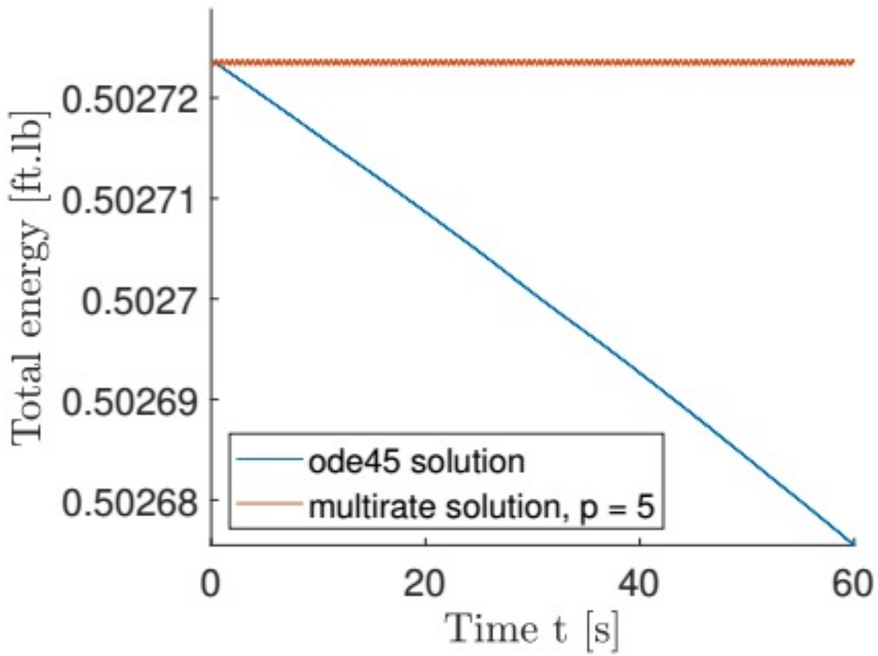


Figure 1: Numerical dissipation of total energy for simulations with $\Delta t = 10^{-4}$ and $t_f = 60$s

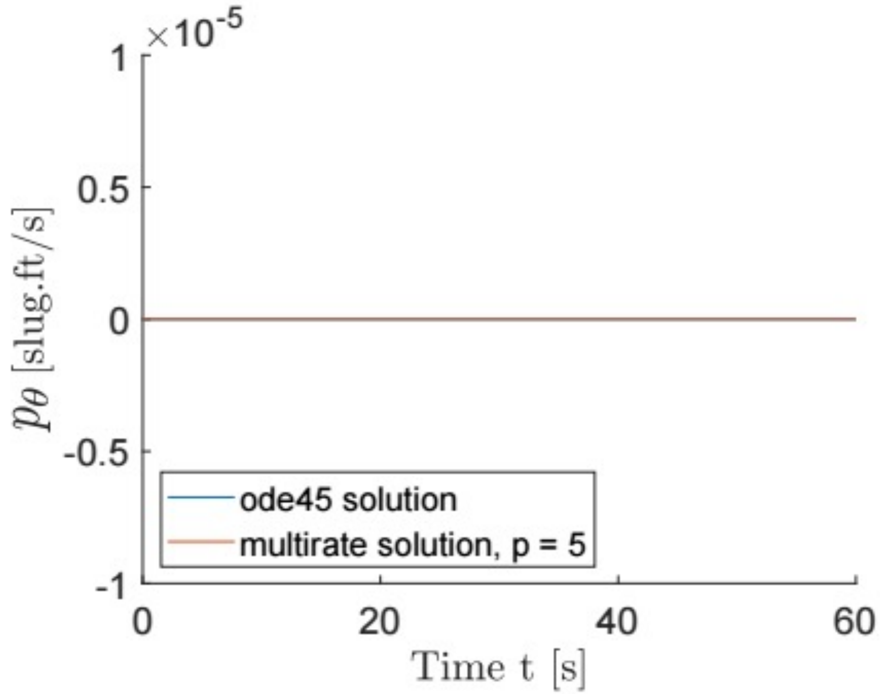


Figure 2: Momentum preservation for simulations with $\Delta t = 10^{-4}$ and $t_f = 60$s

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = M_{\theta\theta}\,\dot{\theta} + (M_{\theta\eta})^T\,\underline{\dot{\eta}}$$

Optimal control problem formulation

$$J(x,u) = \int_{t_0}^{t_f} C(\underline{x}(t), u(t))\, dt = \frac{1}{2}\int_{t_0}^{t_f} [\, \underline{x}(t)^T \mathbf{W} \underline{x}(t) + u(t)^2 \,] dt$$

$$\text{subject to} \quad \begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\tau(t) \\ \underline{q}(t_0) = \mathbf{E}^{-1}\,\underline{\xi}_{t_0}, \quad \underline{\xi}_{t_0} = [0, ..., 0]^T \\ \underline{q}(t_f) = \mathbf{E}^{-1}\,\underline{\xi}_{t_f}, \quad \underline{\xi}_{t_f} = [\theta_{t_f}, 0, ..., 0]^T \\ \dot{\underline{q}}(t_0) = \dot{\underline{q}}(t_f) = [0, ..., 0]^T \end{cases}$$

$$\text{where} \quad \underline{x}(t) = \begin{bmatrix} \underline{q}(t) \\ \dot{\underline{q}}(t) \end{bmatrix}, \;\; u(t) = \tau(t), \;\; \mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{\Lambda} & 0 \end{bmatrix}, \;\; \mathbf{B} = \begin{bmatrix} 0 \\ \mathbf{E}^T\mathbf{D} \end{bmatrix}, \;\; \mathbf{W} = \mathbf{I}$$

Example solution

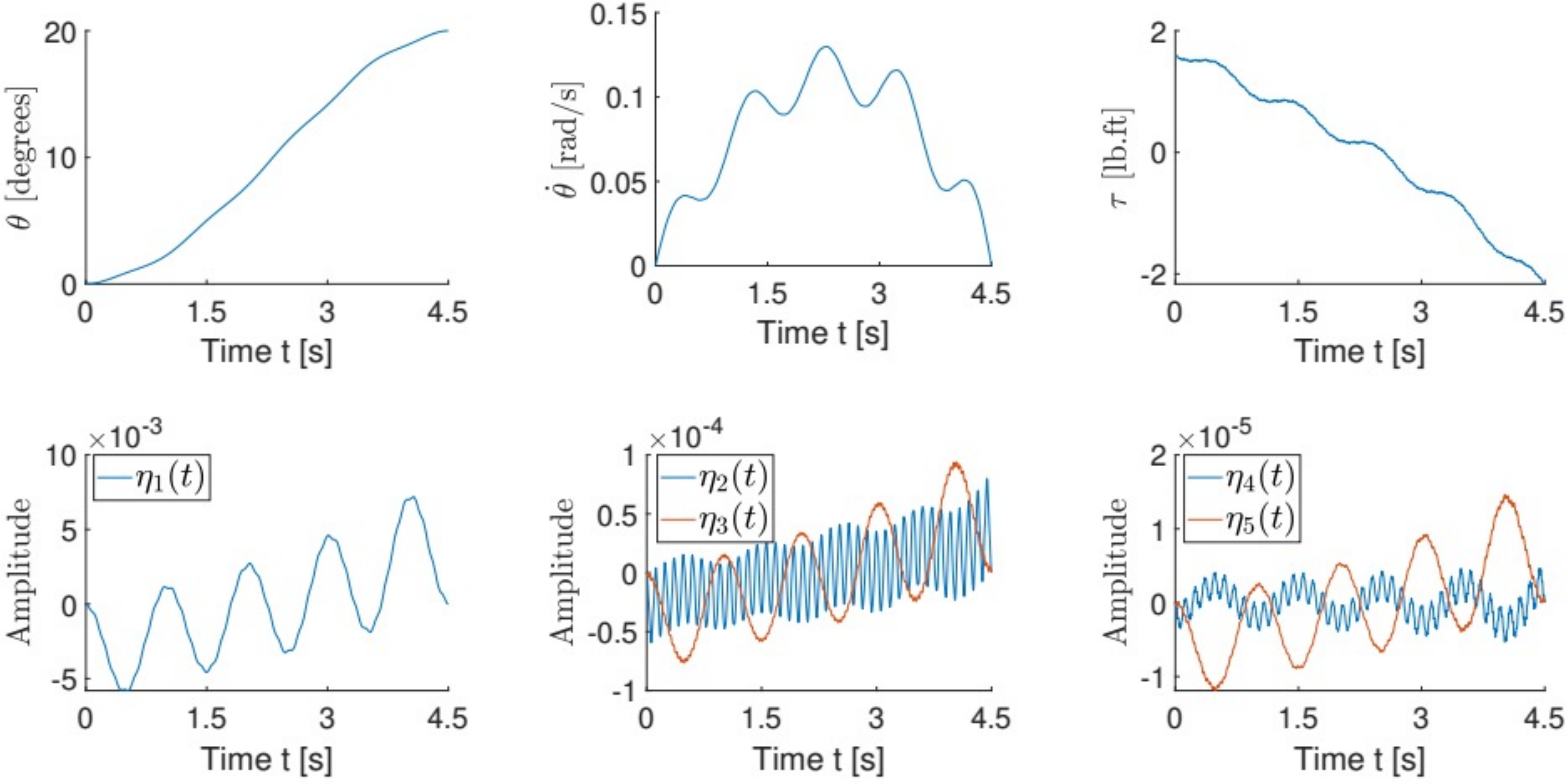
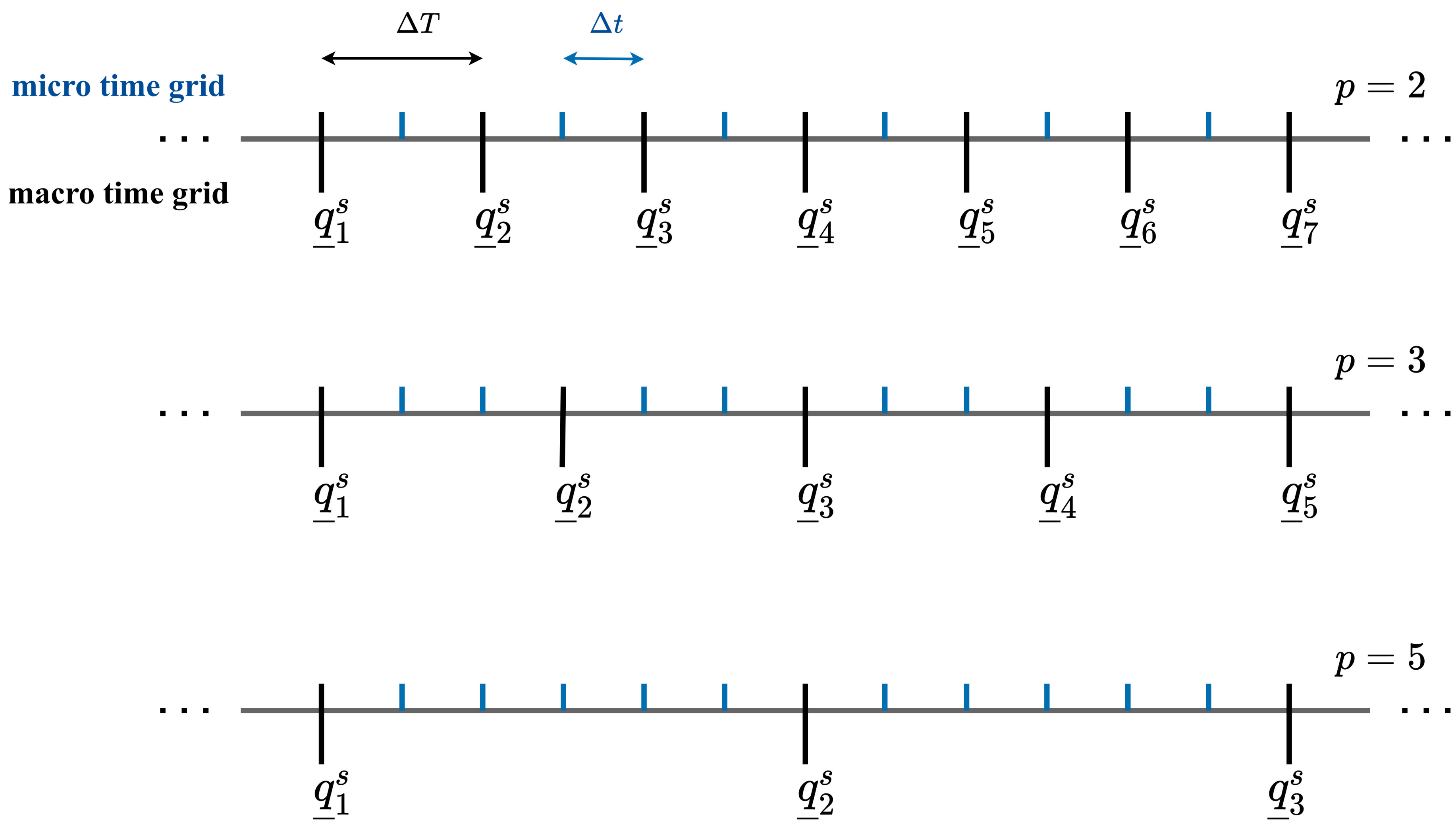


Figure 3: OCP solution with Multirate DMOC for $\theta_{t_f} = 20°$, $t_f = 4.5s$, $\Delta t = 10^{-3}$ and $p = 5$

Problem size

$$p = \frac{\text{macro time step}}{\text{micro time step}} = \frac{\Delta T}{\Delta t}$$



Problem size

$n_{eq\,con}$ - number of equality constraints

$n_{slow\,var}$ - number of optimization variables resulting from discretization on the macro grid

$n_{fast\,var}$ - number of optimization variables resulting from discretization on the micro grid

$n_{total\,var} = n_{slow\,var} + n_{fast\,var}$

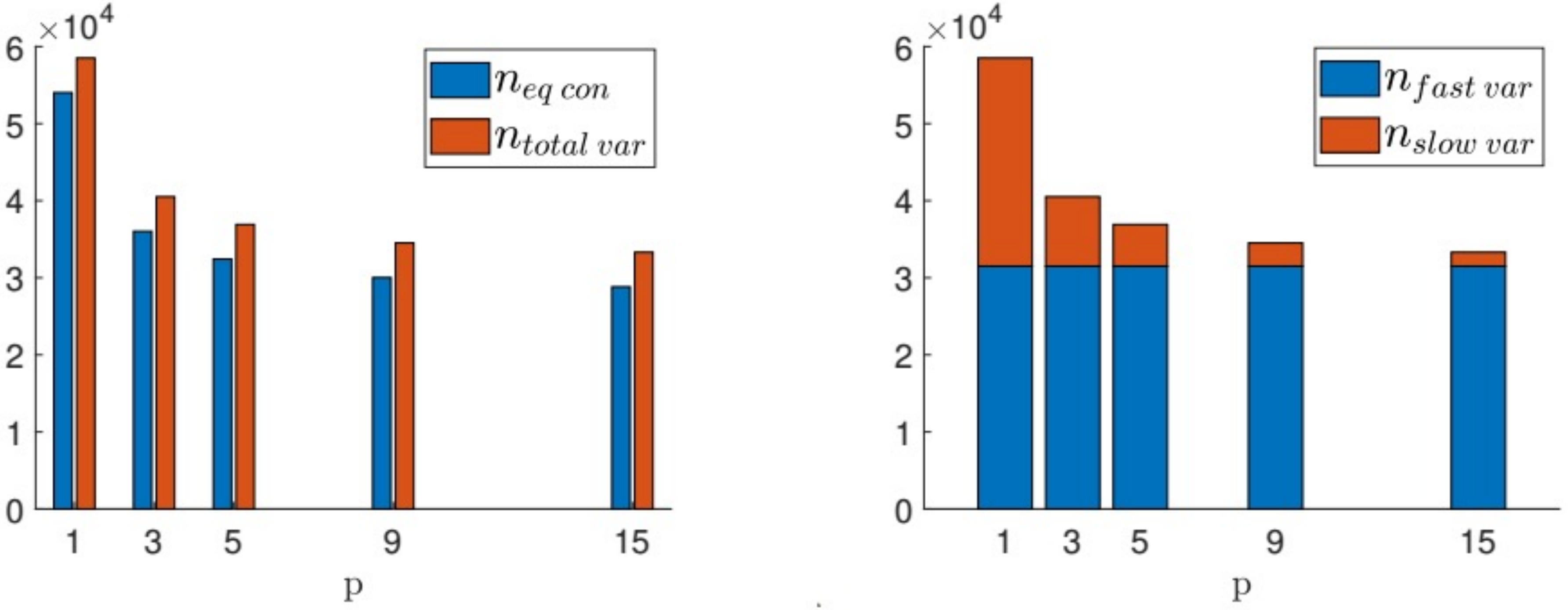


Figure 4: Size of OCP based on Multirate DMOC for a simulations with $\Delta t = 10^{-3}$ and $t_f = 4.5s$

Main result- the trade-off

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$$e^{rel}_{\underline{\xi}}(\underline{\xi}_d, \underline{\xi}) = \frac{\max_{k=0,\ldots,n_s}(\|\underline{\xi}_k - \underline{\xi}(t_k)\|_\infty)}{\max_{k=0,\ldots,n_s}(\|\underline{\xi}(t_k)\|_\infty)}$$

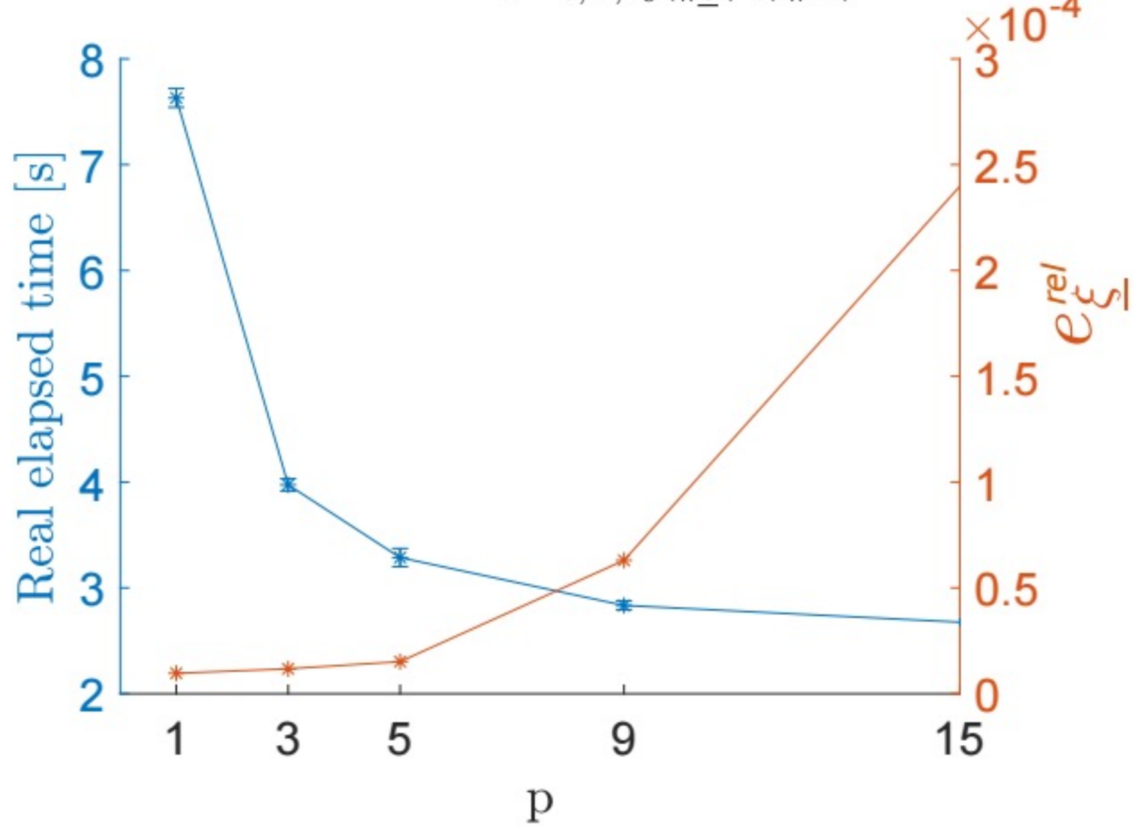

Figure 5: Mean computational time with standard deviation and relative error in $\underline{\xi}$ versus p for a constant micro time step of 10^{-3}, $t_f = 4.5s$ and $\theta_{t_f} = 20°$

Main result- the trade-off

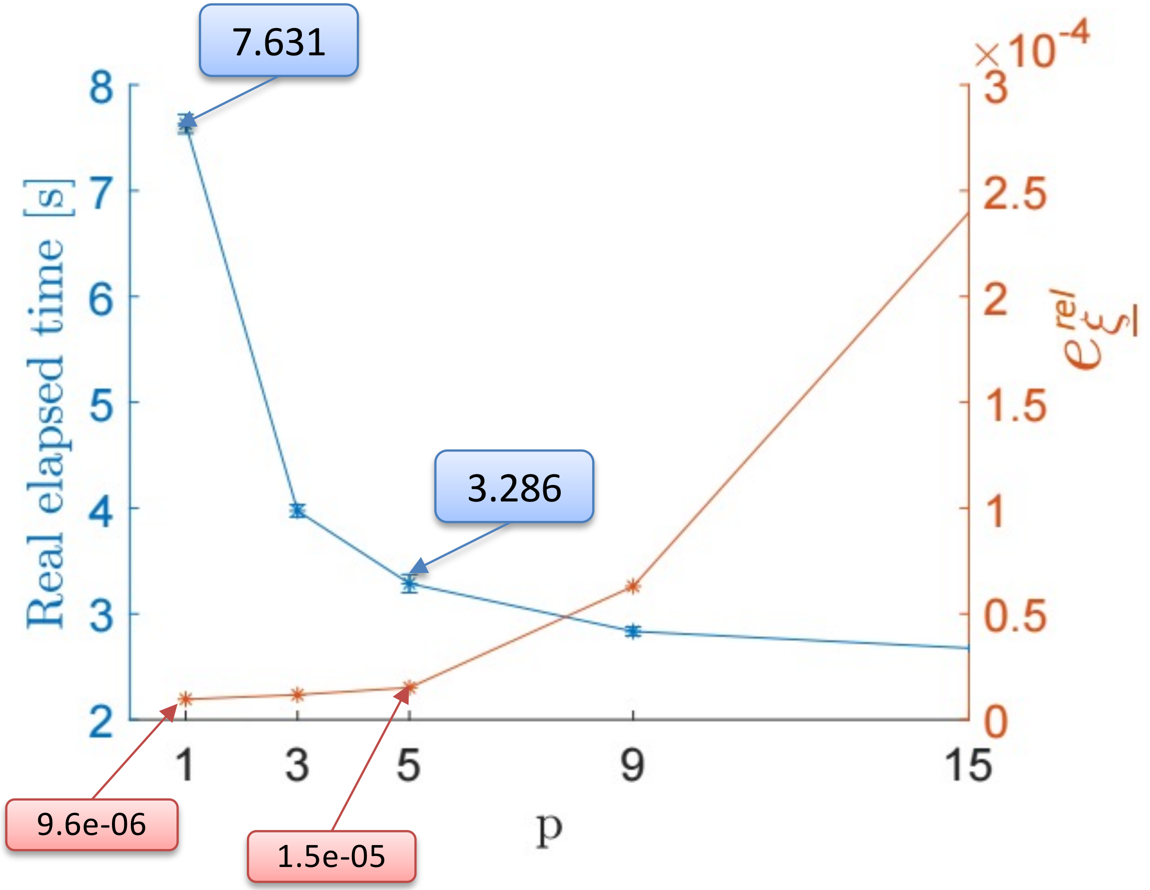

Figure 5: Mean computational time with standard deviation and relative error in $\underline{\xi}$ versus p for a constant micro time step of 10^{-3}, $t_f = 4.5s$ and $\theta_{t_f} = 20°$

Further customization

$\underline{q}^s \in R^{r\times 1}$	r - degrees of freedom of the slow subsystem
$\underline{q}^f \in R^{(N+1-r)\times 1}$	so far in the examples $r = 3$, $N = 5$

$$n_{total\ var} = n_{slow\ var} + n_{fast\ var}$$

$$n_{slow\ var}(p, r, N, t_f, \Delta t) = 2\,r\left(\frac{t_f}{p\,\Delta t} + 1\right)$$

$$n_{fast\ var}(r, N, t_f, \Delta t) = 2\,(N+1-r)\left(\frac{t_f}{\Delta t} + 1\right) + \frac{t_f}{\Delta t}$$

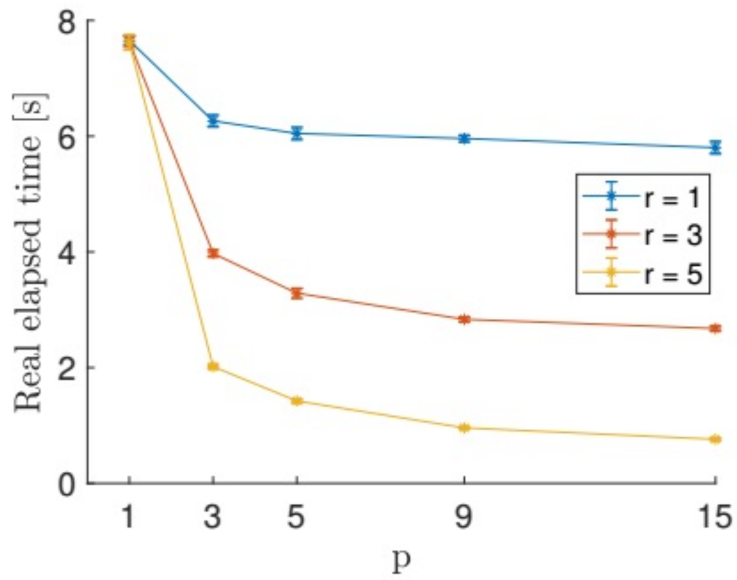

Figure 6: Mean computational time with standard deviation versus p for a constant micro time step of 10^{-3}, $t_f = 4.5s$ and $\theta_{t_f} = 20°$

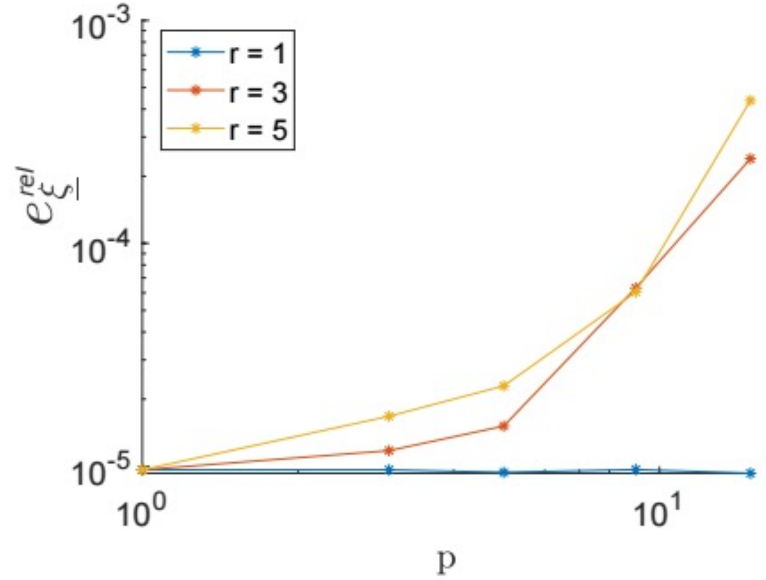

Figure 7: Relative error in $\underline{\xi}$ versus p for a constant micro time step of 10^{-3}, $t_f = 4.5s$ and $\theta_{t_f} = 20°$

Conclusions

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- **Multirate Discrete Mechanics and Optimal Control (Multirate DMOC):**

  High fidelity simulation at a reduced computational cost

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Flexibility to tailor the method to the time scales of the problem and obtain the required fidelity at a reduced computational cost!

- **Future work –** methods for obtaining optimal p and r and extending the work to models including kinematic nonlinearities and dissipation effects

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THANK YOU FOR YOUR ATTENTION!

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Main references:

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