Safe adaptive NMPC using ellipsoidal tubes

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What are we trying to do?

Real applications (aerospace, automotive, biomedical, process control...) involve

- nonlinear dynamics
- unknown disturbance inputs and uncertain model parameters
- constraints on states and control inputs

... and typically need

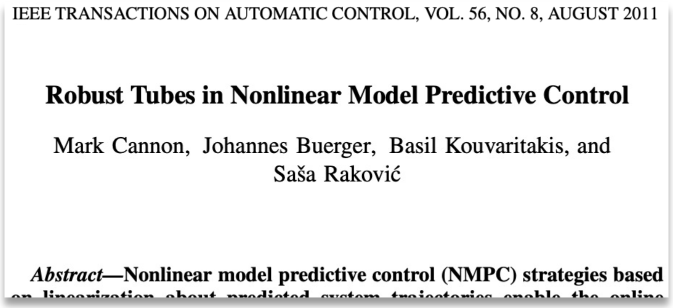
- reliable, scalable, convex problem formulations
- robust control strategies allowing online parameter learning

... and control algorithms that are easy to understand and implement!

What are we trying to do?

This paper builds on concepts from:

- **Robust tubes in nonlinear model predictive control**
  M. Cannon, J. Buerger, B. Kouvaritakis, and S. V. Rakovic
  IEEE Trans. Automatic Control, Vol. 56, no. 8, pp. 1942–1947, 2011


IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 56, NO. 8, AUGUST 2011

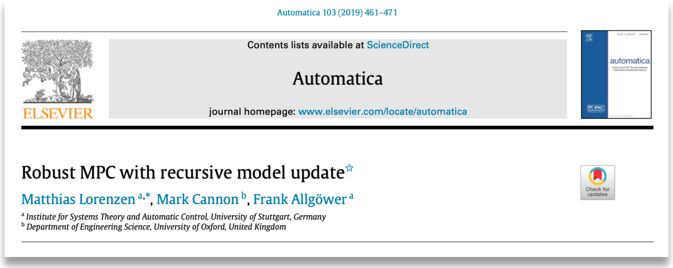
Robust Tubes in Nonlinear Model Predictive Control

Mark Cannon, Johannes Buerger, Basil Kouvaritakis, and Saša Raković

***Abstract*—Nonlinear model predictive control (NMPC) strategies based**

→ Uses ellipsoidal tubes to robustly bound predicted trajectories

- **Robust MPC with recursive model update**
  M. Lorenzen, M. Cannon, and F. Allgower
  Automatica, Vol. 103, pp. 467–471, 2019


Automatica 103 (2019) 461–471

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Robust MPC with recursive model update[☆]

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→ Combines robust predictive control with set-membership estimation

What are we trying to do?

- System model:

$$x_{t+1} = f(x_t, u_t; \theta) + w_t,\ x_t \in \mathcal{X},\ u_t \in \mathcal{U},\ w_t \in \mathcal{W},\ t = 0, 1, \dots$$

  - polytopic state and control constraint sets $\mathcal{X}$, $\mathcal{U}$, and disturbance set $\mathcal{W}$
  - unknown parameters $(\theta_1, \dots, \theta_n) = \theta \in \Theta_0$ appear linearly:

$$f(x_t, u_t, \theta) = f_0(x_t, u_t) + \sum_{i=1}^{n_\theta} \theta_i f_i(x_t, u_t)$$

- Control problem:

minimize $\displaystyle\sum_{t=0}^{\infty} \max_{\mathbf{w} \in \boldsymbol{\mathcal{W}}} \|x_t\|_Q^2 + \|u_t\|_R^2$ subject to $\mathbf{u} \in \boldsymbol{\mathcal{U}},\ \mathbf{x} \in \boldsymbol{\mathcal{X}}$

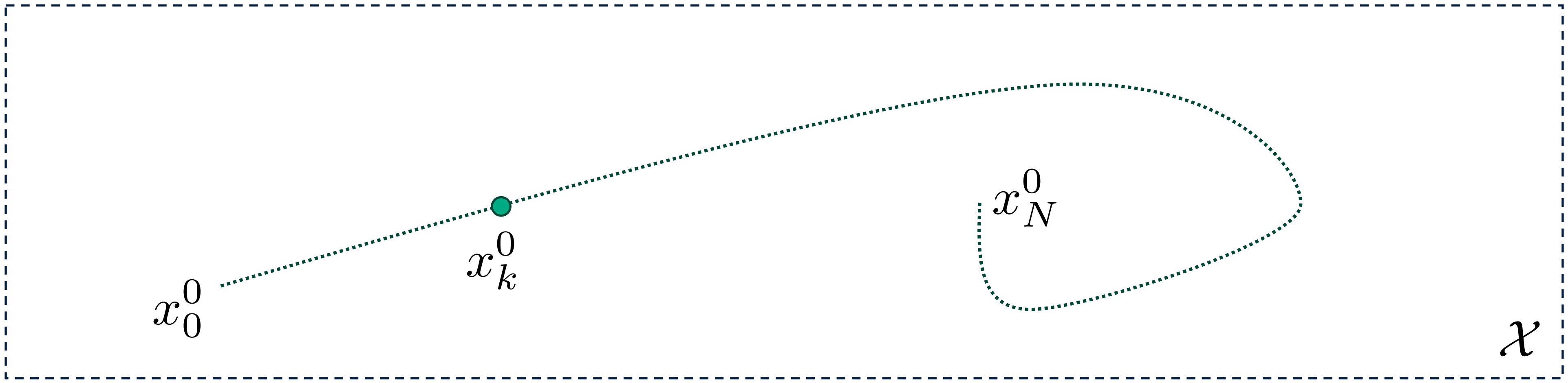
learn $\Theta_t \subseteq \Theta_{t-1} \subseteq \ \cdots \ \subseteq \Theta_0$

Linearisation error bounds

- Initial $\mathbf{x}^0 = \{x_0^0, \ldots, x_N^0\}$, $\mathbf{v}^0 = \{v_0^0, \ldots, v_{N-1}^0\}$ and $\theta^0 \in \Theta$ satisfy

$$x_{k+1}^0 = f_K(x_k^0, v_k^0; \theta^0), \;\; k = 0, \ldots, N-1$$

where $u_k = Kx_k + v_k$, $f_K(x, v; \cdot) = f(x, Kx + v; \cdot)$



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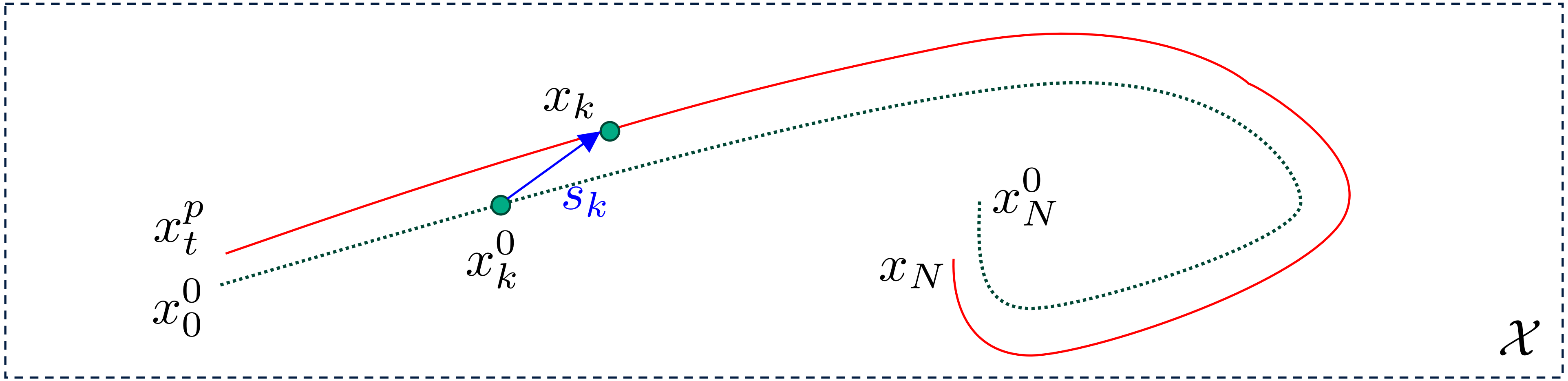
- Perturbations: $x_k = x_k^0 + s_k$, $u_k = Kx_k + v_k^0 + v_k$,



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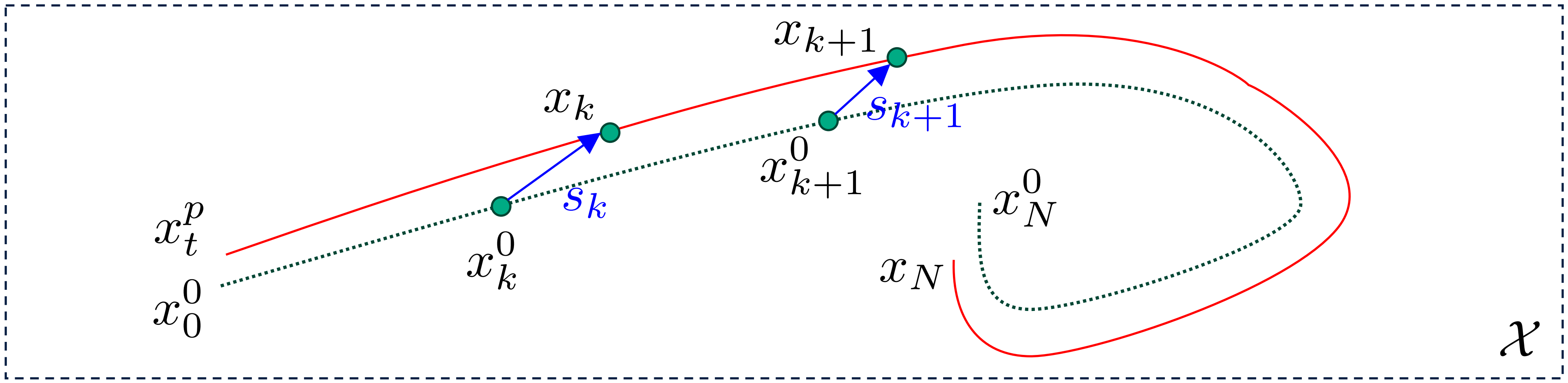
$$x_{k+1}^0 + s_{k+1} = f_K(x_k^0, v_k^0; \theta^0) + \delta_k^0 + \Phi_k s_k + B_k v_k + \delta_k^1 + w_k$$



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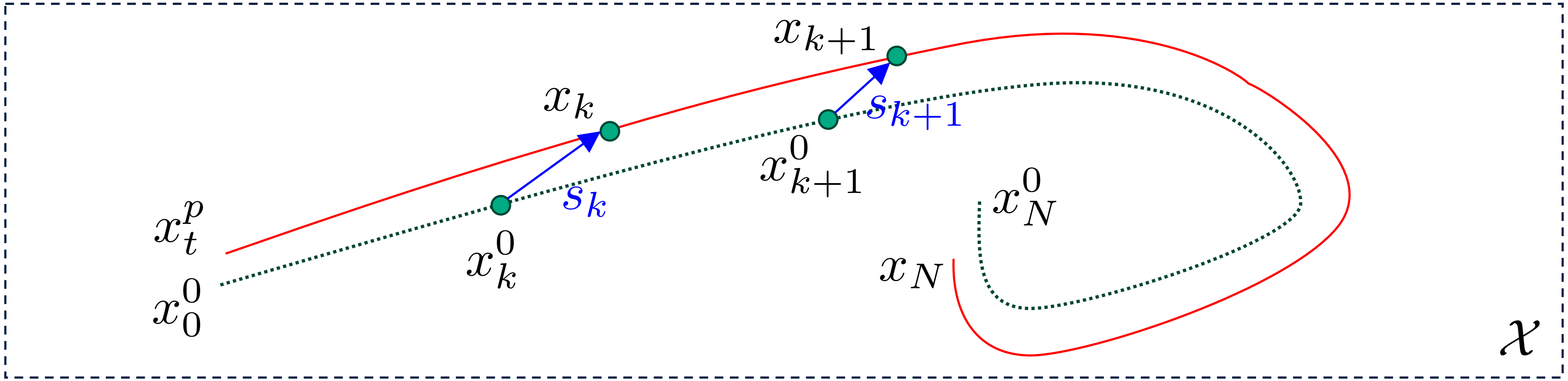
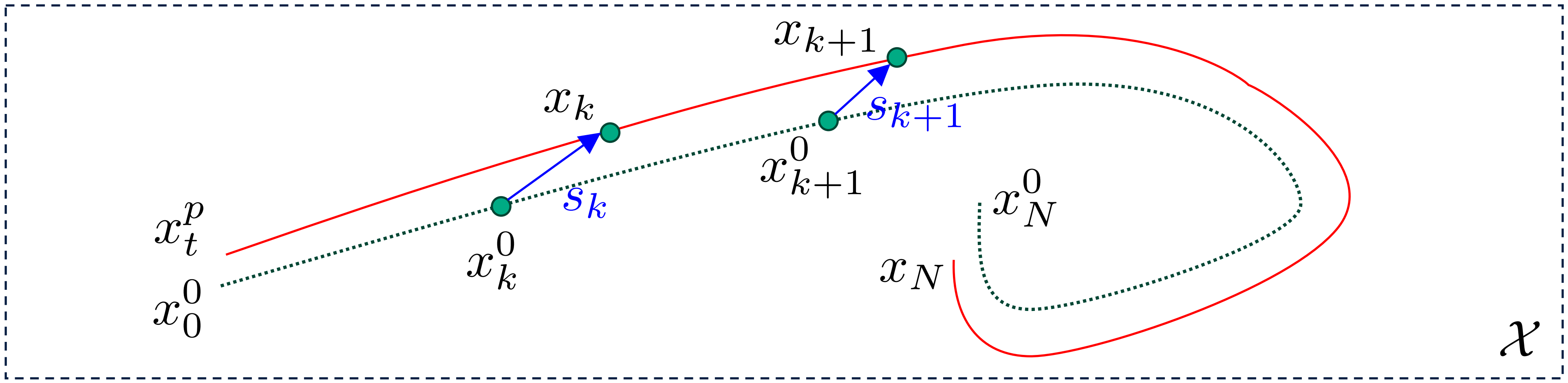
$\Phi_k = \nabla_x f_K(x_k^0, v_k^0; \theta^0)$, $B_k = \nabla_v f_K(x_k^0, v_k^0; \theta^0)$



Linearisation error bounds

$$x^0_{k+1} + s_{k+1} = f_K(x^0_k, v^0_k; \theta^0) + \delta^0_k + \Phi_k s_k + B_k v_k + \delta^1_k + w_k$$



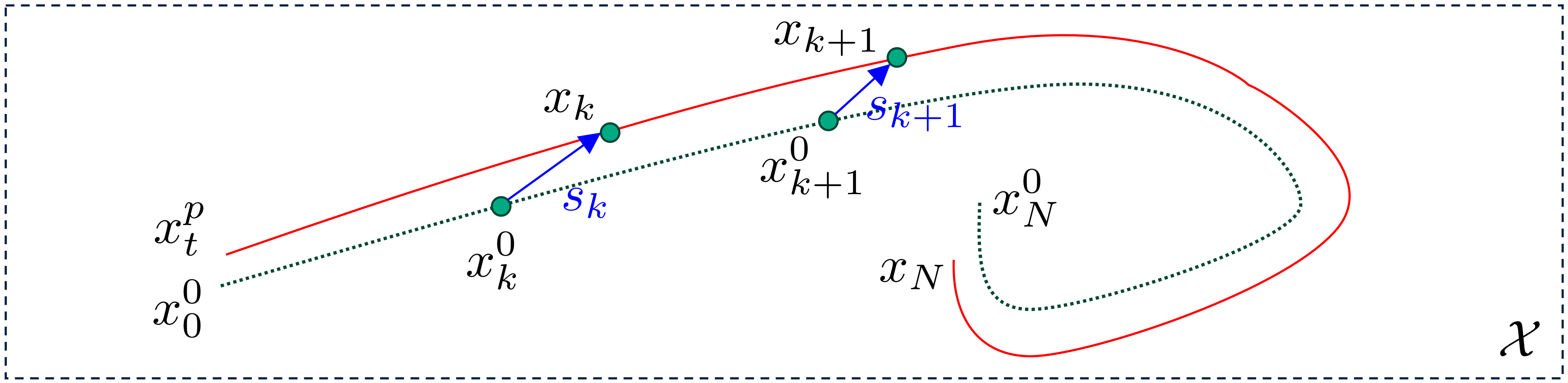
Linearisation error bounds

$$x_{k+1}^0 + s_{k+1} = f_K(x_k^0, v_k^0; \theta^0) + \boxed{\delta_k^0} + \Phi_k s_k + B_k v_k + \delta_k^1 + w_k$$

- $\delta_k^0 = f_K(x_k^0, v_k^0, \theta) - f_K(x_k^0, v_k^0, \theta^0) = f_K(x_k^0, v_k^0, \theta - \theta^0)$

  $\in \mathcal{W}^0(x_k^0, v_k^0, \theta^0, \Theta) = \mathrm{co}\{\delta_k^{0,(q)},\ q \in \mathbb{N}_{\nu_\theta}\}$

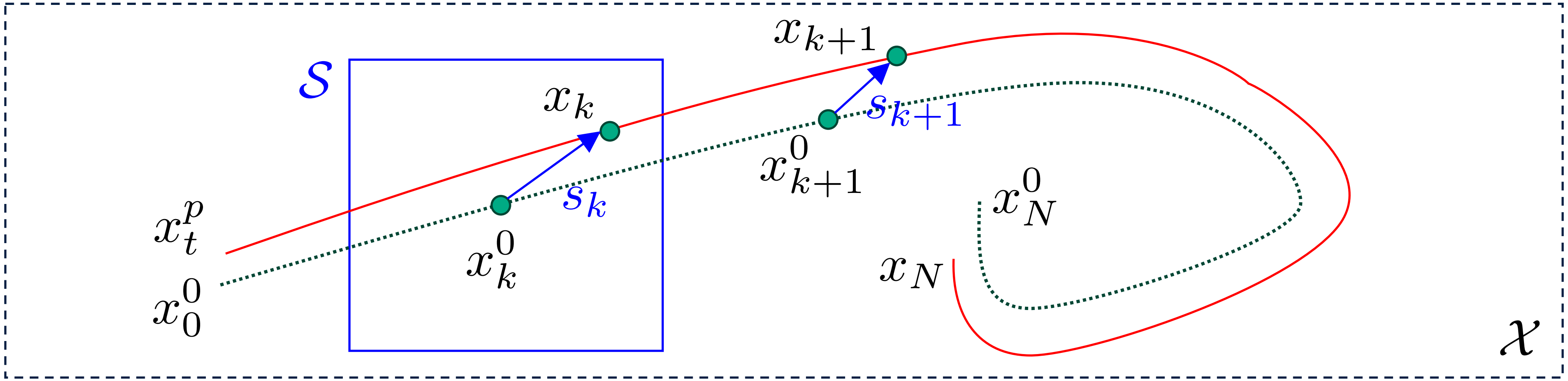
- $\delta_k^{0,(q)}$ determined by the vertices of Θ



Linearisation error bounds

$$x^0_{k+1} + s_{k+1} = f_K(x^0_k, v^0_k; \theta^0) + \delta^0_k + \Phi_k s_k + B_k v_k + \boxed{\delta^1_k} + w_k$$

- Impose constraints $s_k \in \mathcal{S}$, $v_k \in \mathcal{V}$
- $\delta^1_k = f_K(x^0_k + s_k, v^0_k + v_k, \theta) - f_K(x^0_k, v^0_k, \theta) - \Phi_k s_k - B_k v_k$
  $\in \mathcal{W}^1(x^0_k, v^0_k, \theta^0, \mathcal{S}, \mathcal{V}, \Theta) = \mathrm{co}\{C^{(j)}_k s_k + D^{(j)}_k v_k,\ j \in \mathbb{N}_{\nu_1}\}$
- $C^{(j)}_k, D^{(j)}_k$ determined by bounds on $\nabla_x f_K, \nabla_v f_K$ over $s \in \mathcal{S}$, $v \in \mathcal{V}$, $\theta \in \Theta$



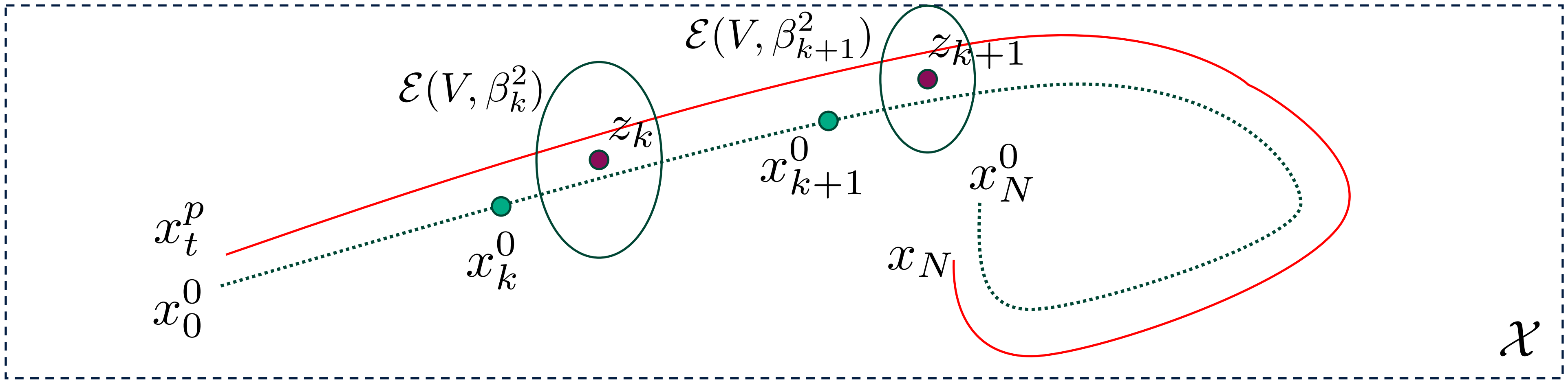
Ellipsoidal tubes

State decomposition: $s_k = z_k + e_k$

nominal dynamics: $z_{k+1} = \Phi_k z_k + B_k v_k$

error dynamics: $e_{k+1} = \Phi_k e_k + w_k + \delta_k^0 + \delta_k^1 \in \mathcal{E}(V, \beta_k^2)$

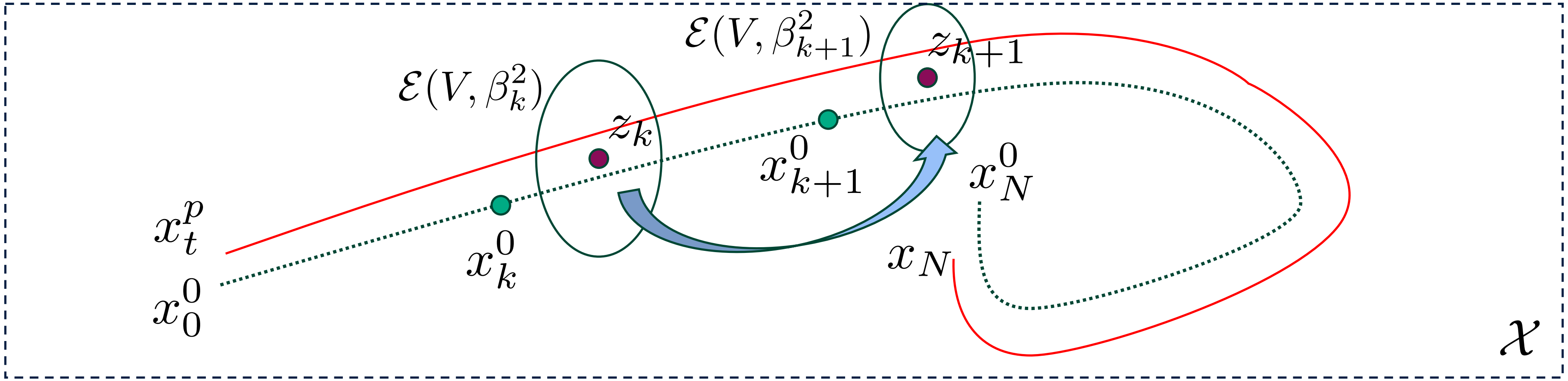
Ellipsoidal bounds: $\mathcal{E}(V, \beta_k^2) = \{e : e^\top V e \leq \beta_k^2\}$



Ellipsoidal tubes

Recursive tube membership condition:

$$\mathcal{E}(V,\beta_{k+1}^2) \ni \Phi_k e + w + \delta^0 + \delta^1,\ \forall w \in \mathcal{W},\ \delta^0 \in \mathcal{W}_k^0,\ \delta^1 \in \mathcal{W}_k^1,\ e \in \mathcal{E}(V,\beta_k^2)$$



Ellipsoidal tubes

Recursive tube membership condition:

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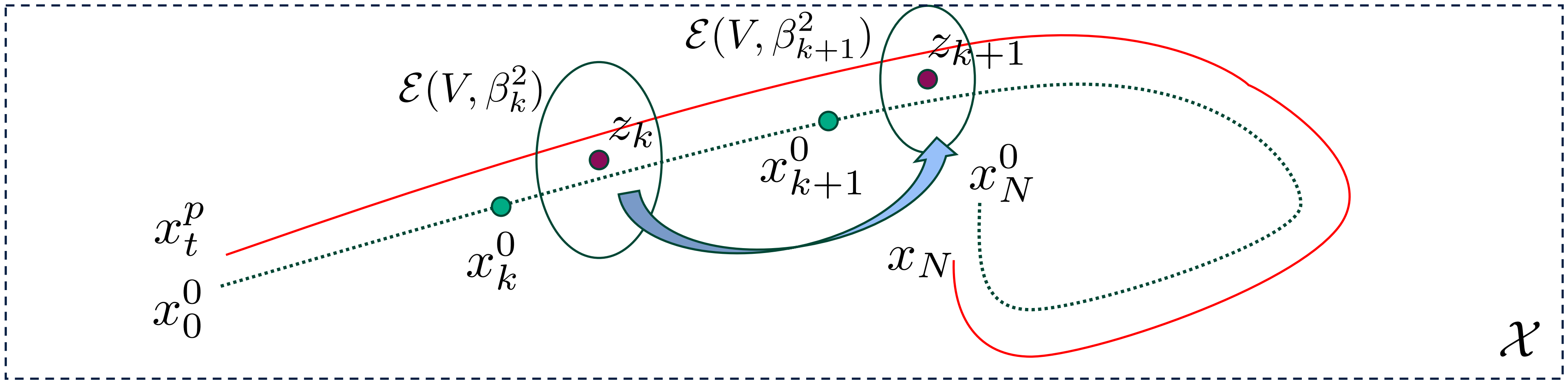
A sufficient condition (triangle inequality):

$$\beta_{k+1} \geq \|C_k^{(j)} z_k + D_k^{(j)} v_k + \delta_k^{0\,(q)}\|_V + \|(\Phi_k + C_k^{(j)})e + w^{(r)}\|_V \qquad \forall e \in \mathcal{E}(V, \beta_k^2)$$



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This holds if

$$\beta_{k+1} \geq (\lambda_k \beta_k^2 + \sigma^2)^{\frac{1}{2}} + \|C_k^{(j)} z_k + D_k^{(j)} v_k + \delta_k^{0\,(q)}\|_V$$

for all $j \in \mathbb{N}_{\nu_1}$, $q \in \mathbb{N}_{\nu_\theta}$, with λ_k defined by

$$\lambda_k = \max_{j \in \mathbb{N}_{\nu_1}, r \in \mathbb{N}_{\nu_r}} \|(\Phi_k + C_k^{(j)}) V^{-\frac{1}{2}}\|_{\Psi^{(r)}}^2$$

where $\Psi^{(r)} = (V^{-1} - w^{(r)} w^{(r)\top} \sigma^{-2})^{-1}$ and σ is a constant

Ellipsoidal tubes

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$\implies$ convex (2nd order cone) condition on the variables β_k, β_{k+1}, z_k, v_k

MPC algorithm at time t

$i \leftarrow 0$, $x_t^p \leftarrow$ plant state at time t

Compute $\mathbf{x}^0$ given $\mathbf{v}^0$
$\Phi_k, B_k \leftarrow$ linearise at $\mathbf{x}^0, \mathbf{v}^0, \theta^0$
Compute λ_k & bounds $\mathcal{W}_k^0, \mathcal{W}_k^1$
Solve SOCP for $\mathbf{v}^*$, $J^{(i)}$

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MPC optimisation (SOCP) at iteration i, time t:

$$(\mathbf{v}^\star, \boldsymbol{\beta}^\star, \mathbf{z}^\star) = \arg \min_{\mathbf{v},\boldsymbol{\beta},\mathbf{z},\mathbf{l}} J_t^{(i)} = \sum_{k=0}^{N} l_k^2$$

subject to, for $k = 0, \dots, N-1$ and all $j \in \mathbb{N}_{\nu_1}$, $q \in \mathbb{N}_{\nu_\theta}$,

$$z_{k+1} = \Phi_k z_k + B_k v_k$$

$$\beta_{k+1} \geq (\lambda_k \beta_k^2 + \sigma^2)^{\frac{1}{2}} + \|C_k^{(j)} z_k + D_k^{(j)} v_k + \delta_k^{0\,(q)}\|_V$$

$$l_k \geq \left(\|x_k^0 + z_k\|_Q^2 + \|K(x_k^0 + z_k) + v_k^0 + v_k\|_R^2\right)^{\frac{1}{2}} + \beta_k \|V^{-\frac{1}{2}}\|_{Q+K^\top RK}$$

$$\mathcal{U} \supset K\big(x_k^0 + z_k + \mathcal{E}(V, \beta_k^2)\big) + v_k^0 + v_k, \quad \mathcal{X} \supset x_k^0 + z_k + \mathcal{E}(V, \beta_k^2)$$

$$\mathcal{V} \ni v_k^0 + v_k, \quad \mathcal{S} \supset z_k + \mathcal{E}(V, \beta_k^2)$$

and initial and terminal conditions

$$\beta_0 \geq \|x_0^0 + z_0 - x_t^p\|_V, \quad \Omega(x_N^0) \ni (\|z_N\|_V, \beta_N), \quad l_N \geq \hat{l}(z_N, \beta_N, x_N^0)$$

and, for iteration $i = 1$,

$$J_t^{(i)} \leq J_{t-1}^{(final)} - \left(\|x_{t-1}\|_Q^2 + \|u_{t-1}\|_R^2 - \hat{\sigma}^2\right)$$

and, for iterations $i > 1$,

$$J_t^{(i)} \leq J_t^{(i-1)}$$

MPC algorithm at time t

$i \leftarrow 0$, $x_t^p \leftarrow$ plant state at time t

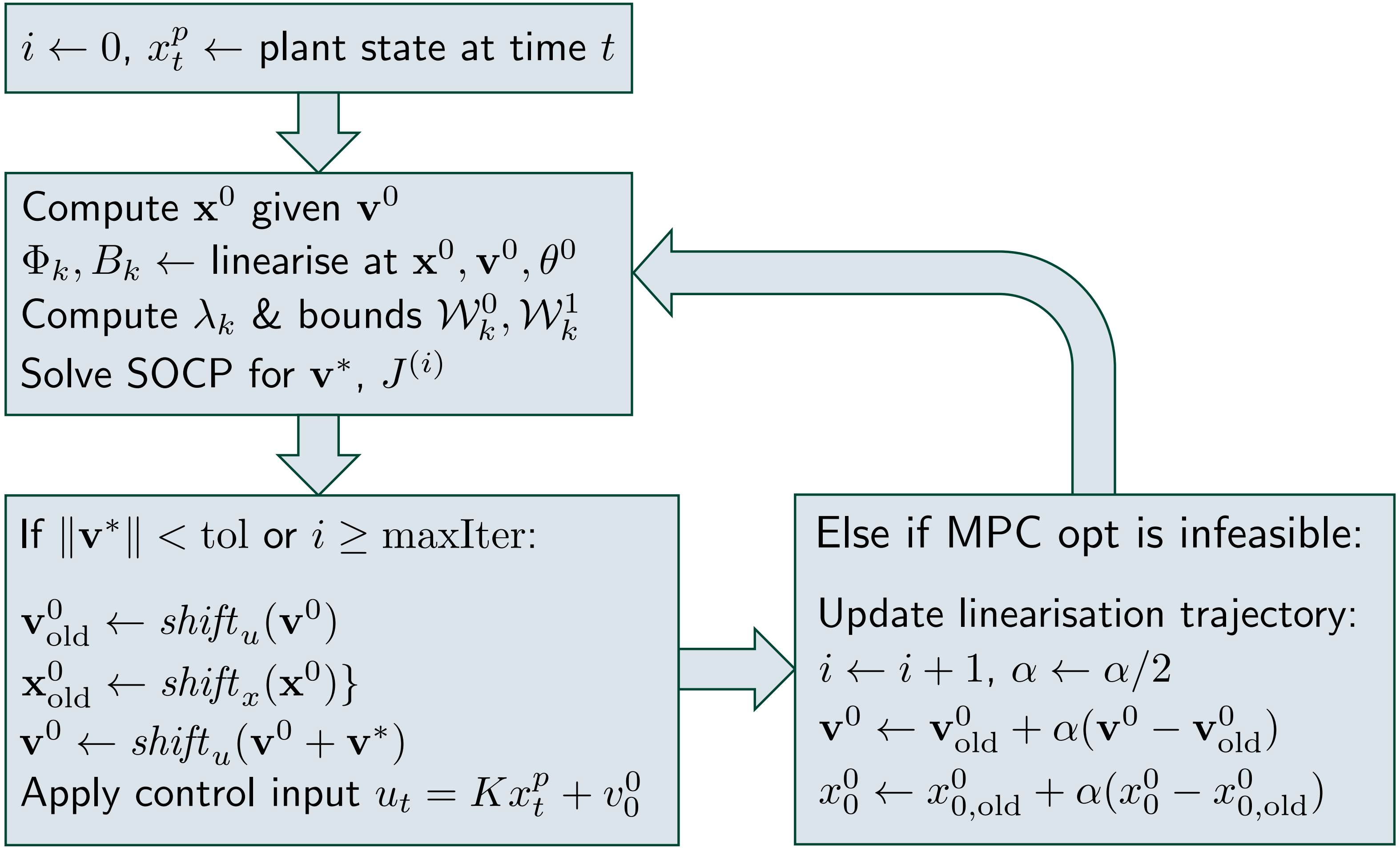
Compute $\mathbf{x}^0$ given $\mathbf{v}^0$
$\Phi_k, B_k \leftarrow$ linearise at $\mathbf{x}^0, \mathbf{v}^0, \theta^0$
Compute λ_k & bounds $\mathcal{W}_k^0, \mathcal{W}_k^1$
Solve SOCP for $\mathbf{v}^*$, $J^{(i)}$

If $\|\mathbf{v}^*\| < \mathrm{tol}$ or $i \geq \mathrm{maxIter}$:

$\mathbf{v}_{\mathrm{old}}^0 \leftarrow \mathit{shift}_u(\mathbf{v}^0)$
$\mathbf{x}_{\mathrm{old}}^0 \leftarrow \mathit{shift}_x(\mathbf{x}^0)\}$
$\mathbf{v}^0 \leftarrow \mathit{shift}_u(\mathbf{v}^0 + \mathbf{v}^*)$
Apply control input $u_t = Kx_t^p + v_0^0$

$$\mathit{shift}_u(\mathbf{v}) \triangleq \{v_1, \ldots, v_{N-1}, 0\}$$
$$\mathit{shift}_x(\mathbf{x}) \triangleq \{x_1, \ldots, x_N, f_K(x_N, 0; \theta^0)\}$$

MPC algorithm at time t



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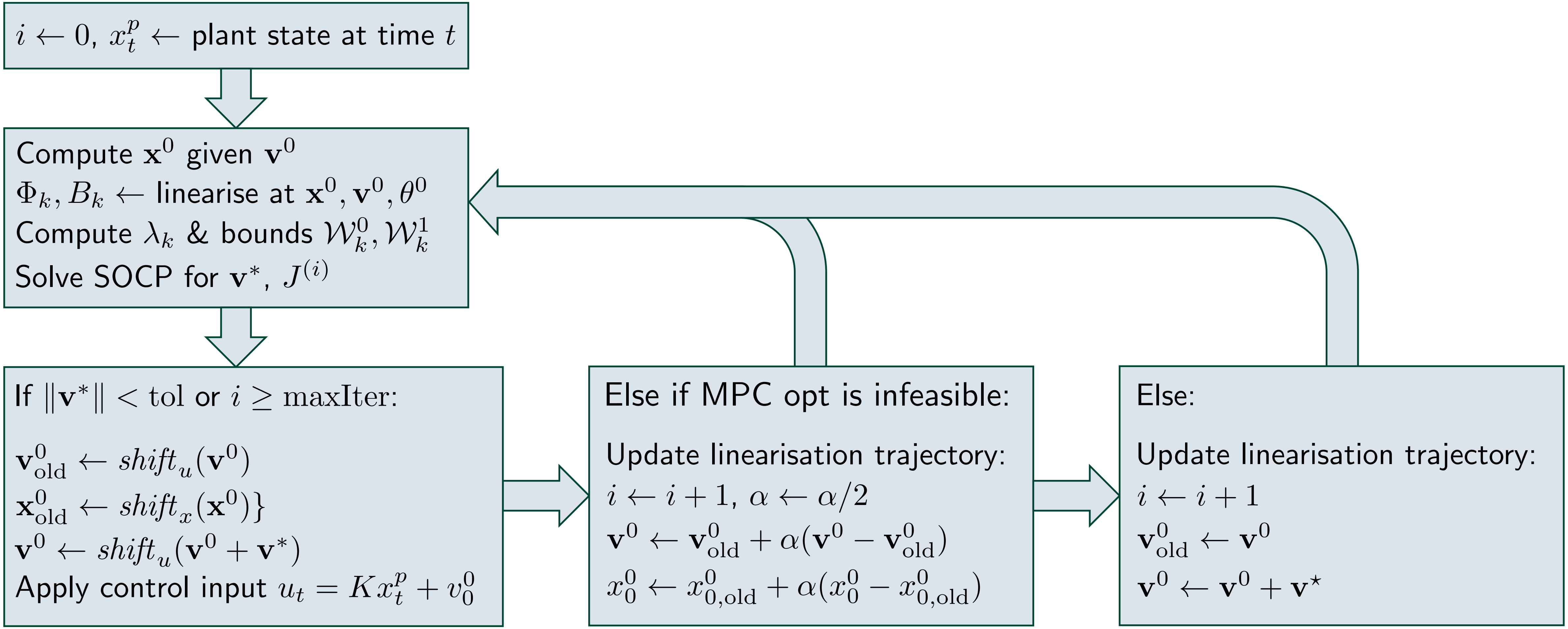

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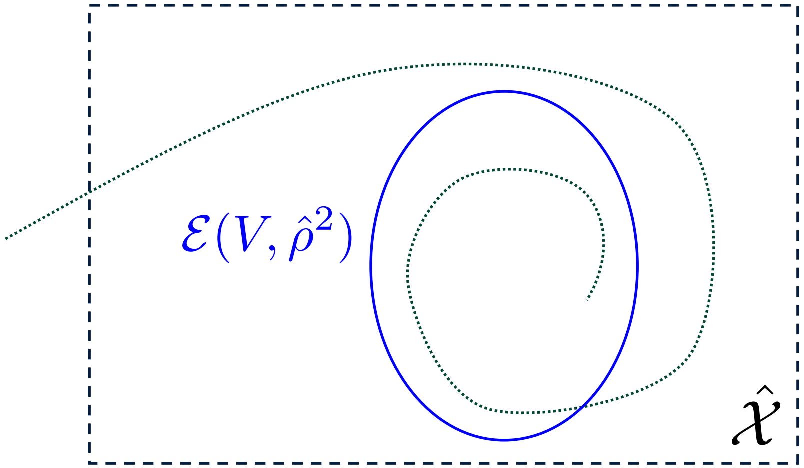
Terminal conditions

Offline: find V, K satisfying, for all $x \in \hat{\mathcal{X}} \subseteq \mathcal{X}$ such that $Kx \in \hat{\mathcal{U}} \subseteq \mathcal{U}$ and for all $\theta \in \Theta_0$,

$$\|x\|_V^2 - \|f(x, Kx, \theta) + w\|_V^2 \geq \|x\|_Q^2 + \|Kx\|_R^2 - \sigma^2$$

e.g. via SDP with LDI model $f(x_k, u_k, \theta) \in \mathrm{co}\{\hat{A}^{(j)} x_k + \hat{B}^{(j)} u_k,\ j \in \mathbb{N}_{\hat{\nu}}\}$

- then $\mathcal{E}(V, \hat{\rho}^2)$ is robust invariant for $\hat{\rho} = \max\{\|x\|_V : x \in \hat{\mathcal{X}}\ \&\ Kx \in \mathcal{U}\}\}$



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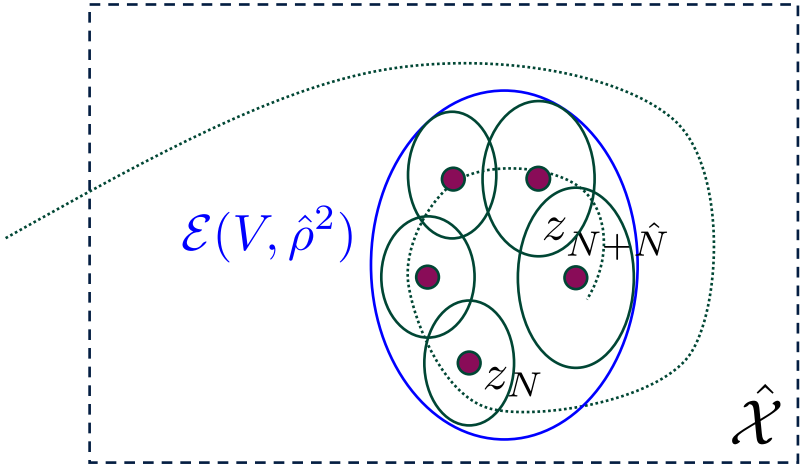
- then $\mathcal{E}(V, \hat{\rho}^2)$ is robust invariant for $\hat{\rho} = \max\{\|x\|_V : x \in \hat{\mathcal{X}}\ \&\ Kx \in \mathcal{U}\}\}$
- ensure $\|x_k^0 + z_k\|_V \leq \hat{\rho}$ for all $k \geq N$

  by defining the terminal set for $(\|z_N\|_V, \beta_N)$ as

$$\Omega(x_N^0) = \Big\{(r, \beta_N) : \beta_N \leq \hat{\rho} - (r + \|x_N^0\|_V)$$
and $\exists \beta_k$ satisfying, for $k = N+1, \ldots, N+\hat{N}$,
$$\beta_k \geq (\hat{\lambda}\beta_{k-1}^2 + \sigma^2)^{\frac{1}{2}} + \hat{\lambda}^{\frac{(k-N-1)}{2}}(r d_0 + d_1),$$
$$\beta_k \leq \hat{\rho} - \hat{\lambda}^{\frac{k-N}{2}}(r + \|x_N^0\|_V)\Big\}$$



Feasibility, stability, and convergence

If the MPC optimisation is feasible at time $t = 0$, then:

1. MPC optimisation is feasible at all times $t \geq 0$ and iterations $i \geq 0$ with $\mathbf{v}^0 = \mathbf{v}^0_{\text{old}}$ (and $x^0_0 = x^0_{0,\text{old}}$ if $i = 0$)

2. Final iteration cost is an ISS Lyapunov function,

$$J^{(final)}_{t+1} - J^{(final)}_{t} \leq -\|x_t\|^2_Q - \|u_t\|^2_R + \bar{\sigma}^2$$

   where $\bar{\sigma} = \gamma\sigma + \gamma\hat{\rho}(d_{\hat{\Phi}} + d_{\Theta}L)$

3. Closed loop trajectories satisfy

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} (\|x_t\|^2_Q + \|u_t\|^2_R) \leq \bar{\sigma}^2$$

Example 1: computation

- Monte Carlo simulations with quadratic nonlinearities:

$$f_0(x,u) = Ax + Bu, \quad f_i(x,u) = \hat{e}_i [x]_{j_i}^2, \ i \in \mathbb{N}_{n_\theta}$$
$$\mathcal{W} = \{B_w \hat{w} : \|\hat{w}\|_\infty \leq 0.01\}$$

  A, B, B_w and $j_i \in \{1, \ldots, \mathbb{N}_x\}$ randomly generated

- true parameter θ^* and Θ_0 randomly chosen

  Θ_t updated using SME with horizon $N_\Theta = 5$

  θ_t^0 defined as the centroid of Θ_t

Example 1: computation

(n_x, n_u, n_θ)	$(2,1,2)$	$(4,2,2)$	$(4,2,4)$	$(6,2,4)$	$(5,2,5)$	$(6,2,6)$	$(8,2,8)$	$(8,4,8)$	$(10,4,10)$	$(12,4,12)$
Variables	48	60	60	62	61	62	64	84	86	88
Equalities	22	44	44	66	55	66	88	88	110	132
Inequalities	57	97	97	117	107	117	137	177	197	217
SOC constraints	294	474	1274	1774	2184	3454	7314	7314	13334	21994
Execution time (s)	0.037	0.119	0.461	0.562	0.802	1.50	4.32	4.23	13.17	50.18

- MPC optimisation via Gurobi and Yalmip (Apple M3 Pro, 36 GB memory)

  Dominant factor is # SOC constraints

  Empirical fit: time per iter $\sim (n_\theta + 1)^{4.2}$ (R^2 value 0.97)

Example 1: computation

(n_x, n_u, n_θ)	$(2,1,2)$	$(4,2,2)$	$(4,2,4)$	$(6,2,4)$	$(5,2,5)$	$(6,2,6)$	$(8,2,8)$	$(8,4,8)$	$(10,4,10)$	$(12,4,12)$
Variables	48	60	60	62	61	62	64	84	86	88
Equalities	22	44	44	66	55	66	88	88	110	132
Inequalities	57	97	97	117	107	117	137	177	197	217
SOC constraints	294	474	1274	1774	2184	3454	7314	7314	13334	21994
Execution time (s)	0.037	0.119	0.461	0.562	0.802	1.50	4.32	4.23	13.17	50.18

- MPC optimisation via Gurobi and Yalmip (Apple M3 Pro, 36 GB memory)

  Dominant factor is # SOC constraints

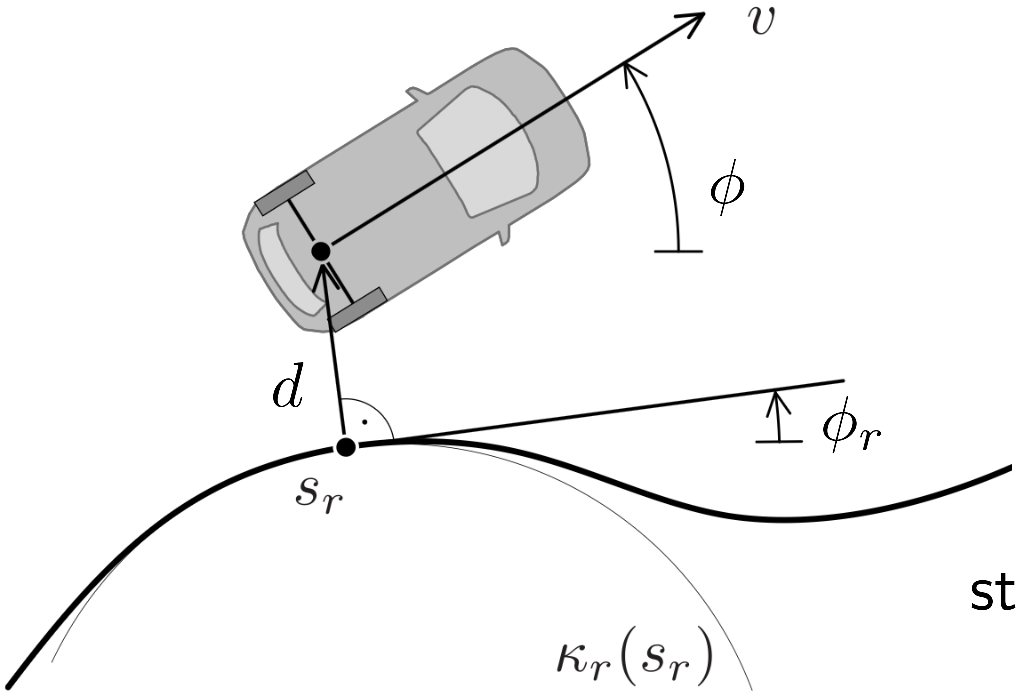
  Empirical fit: time per iter $\sim (n_\theta + 1)^{4.2}$ (R^2 value 0.97)

- For higher order polynomial nonlinearities time per iter is approximately 50 sec

  with

n_θ:	12	24	28	34	42	56
n_x:	12	6	5	4	3	2

Example 2: automated driving



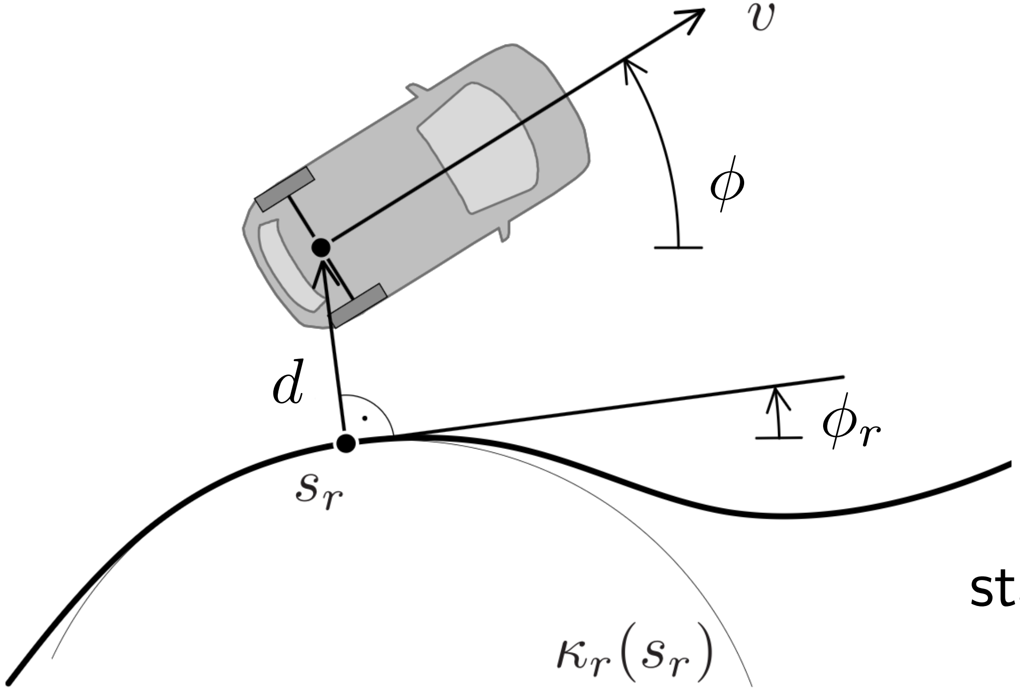
Continuous time model:

$$\dot{v}^\delta = u_1 + \theta_1 + w_1$$
$$\dot{d} = (v^r + v^\delta)\sin(\phi)$$
$$\dot{\phi} = (v^r + v^\delta)\kappa$$
$$\dot{\kappa} = u_2 + \theta_2 + w_2$$

state $x = \begin{bmatrix} v^\delta \\ d \\ \phi \\ \kappa \end{bmatrix}$

- v^δ: velocity relative to reference v^r
- d: lateral distance from reference
- ϕ: angle of path
- κ: curvature of path

Example 2: automated driving



Continuous time model:

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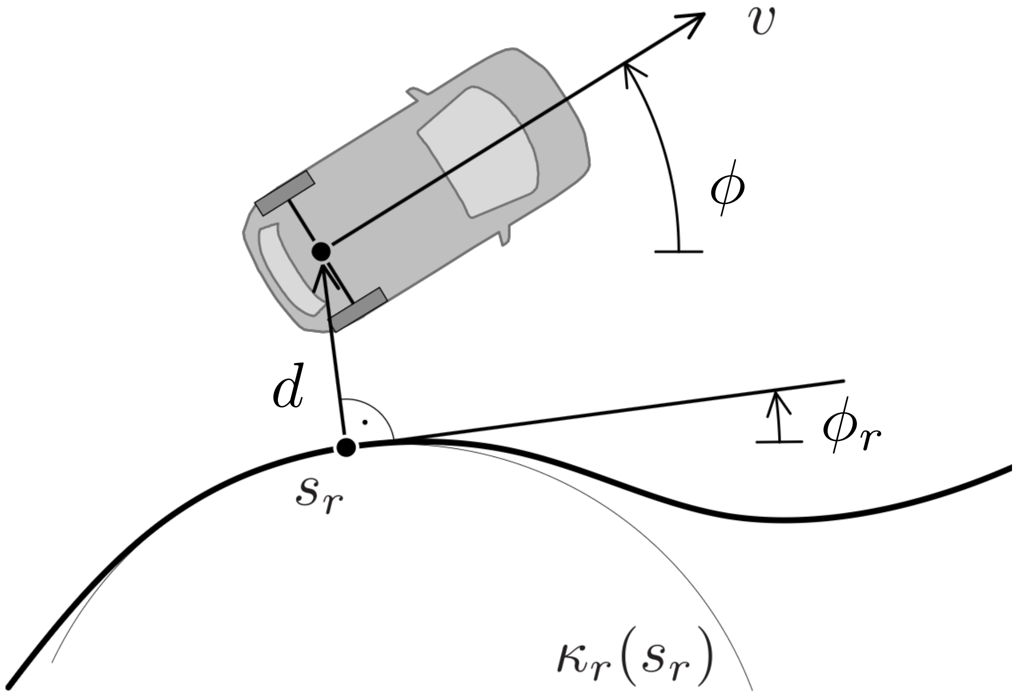
state $x = \begin{bmatrix} v^\delta \\ d \\ \phi \\ \kappa \end{bmatrix}$
- v^δ: velocity relative to reference v^r
- d: lateral distance from reference
- ϕ: angle of path
- κ: curvature of path

control input $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
- u_1: longitudinal thrust
- u_2: lateral thrust

(w_1, w_2): bounded disturbances
(θ_1, θ_2): unknown parameters

Example 2: automated driving



Discrete time sampling interval $T_s = 0.1$:

$$f_0(x_t, u_t) = \begin{bmatrix} v_t^\delta + T_s u_{t,1} \\ d_t + (v_t^r + v_t^\delta)\sin(\phi_t) T_s \\ \phi_t + (v_t^r + v_t^\delta)\kappa_t T_s \\ \kappa_t + T_s u_{t,2} \end{bmatrix}$$

$$f_1(x_t, u_t) = T_s \hat{e}_1,$$

$$f_2(x_t, u_t) = T_s \hat{e}_4,$$

(w_1, w_2): bounded disturbances
(θ_1, θ_2): unknown parameters

control input $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ u_1: longitudinal thrust, u_2: lateral thrust

Example 2: automated driving

- Control and disturbance sets:

$$\mathcal{U} = \{(u_1, u_2) : |u_1| \leq 5,\ |u_2| \leq 3\}$$
$$\mathcal{W} = \{(w_1, w_2) : \|w\|_\infty \leq 0.05\}$$

- cost weights: $Q =$ diag$\{1, 0.2, 2, 1\}$, $R =$ diag$\{0.008, 0.004\}$,

  prediction horizon: $N = 15$ time steps

  reference velocity $v^r = 10$ ms

Example 2: automated driving

- True parameter: $\theta^* = (0.02,\, 0.02)$

  initial parameter set estimate: $\Theta_0 = \{\theta : \|\theta\|_\infty \leq 0.04\}$

  Θ_t updated using SME with horizon $N_\Theta = 5$

  $\theta_t^0 =$ centroid of Θ_t

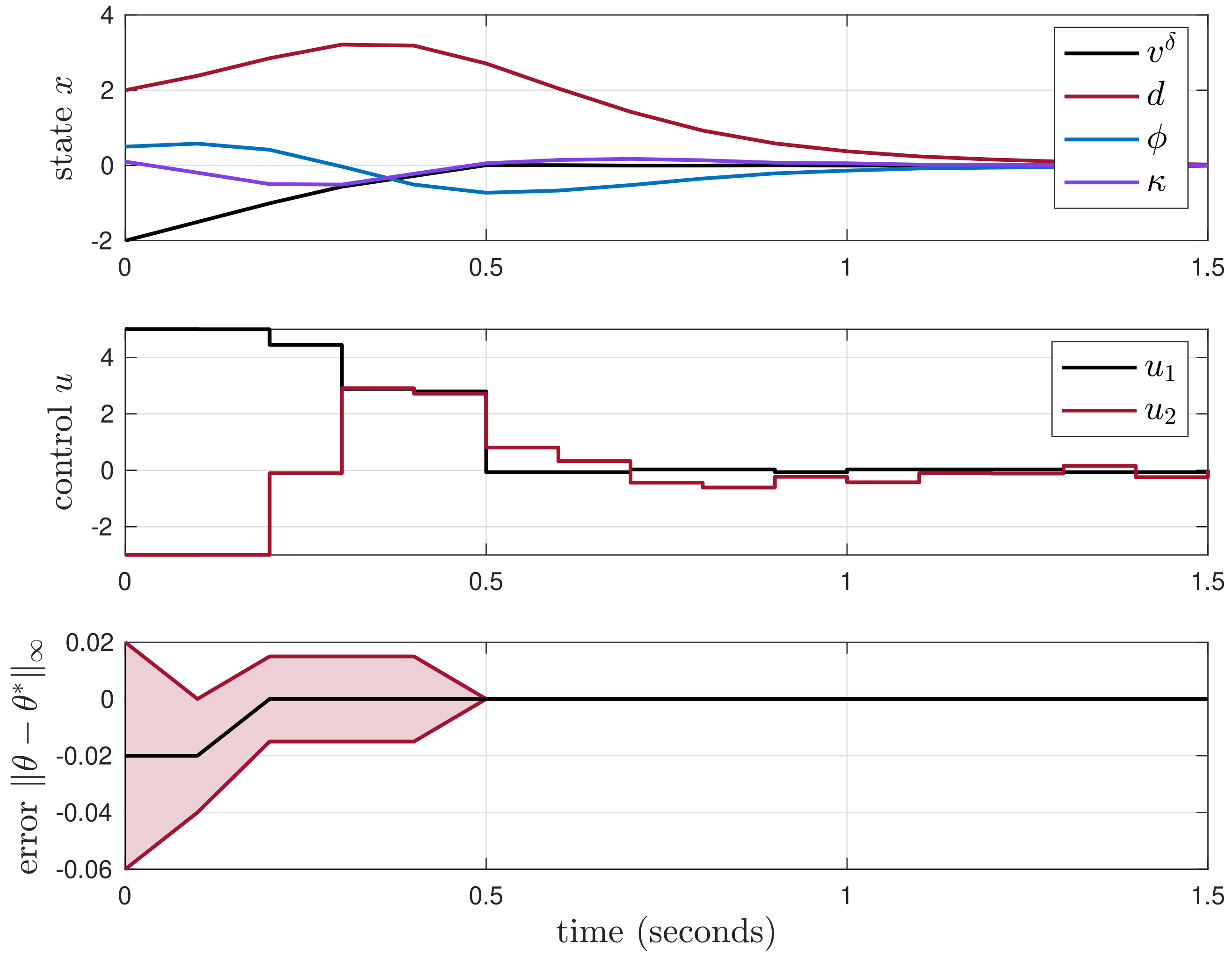
- terminal set computed using LDI model, valid for $\|x\|_\infty \leq 0.5$ and Θ_0

$$\implies \begin{cases} \sigma = 0.03,\ \hat{\lambda} = 0.87,\ \gamma = 3.96,\ \hat{\rho} = 0.39 \\ \hat{N} = 13 \text{ and } \hat{\sigma} = 0.206 \end{cases}$$

- state perturbation bounds $\mathcal{S} = \{s : \|s\|_\infty \leq 0.25\}$

  no control perturbation bounds needed $\implies \mathcal{V} = \mathbb{R}^2$

Example 2: automated driving



- Constraints on u_1, u_2 are active initially
- Parameter error converges within 5 time steps
- Disturbances are rejected

Summary

We propose a robust nonlinear MPC strategy with online model learning based on ellipsoidal tubes and set membership parameter estimation

Convex online optimization; efficient scaling with state, control input, and parameter vector dimensions

Guarantees of recursive feasibility, constraint satisfaction, stability (ISS)

Extensions:

- time-varying ellipsoidal tube shapes and local feedback gains
- convexification via differences of convex functions