C21 Model Predictive Control

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4 lectures

Hilary Term 2023

Department of Engineering Science
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Lecture 1

Introduction

Organisation

- 4 lectures – LR2, weeks 3 & 4
  Monday at 15.00 & Friday at 12.00
  recordings available on Canvas
- Examples class – LR3, week 5
  Friday at 14:00, 16:00 or 17:00
  sign up on Canvas

Course outline

1. Introduction to predictive control
2. Prediction and optimization
3. Closed loop properties
4. Disturbances and integral action
5. Robust tube MPC

Books

- J.M. Maciejowski, *Predictive control with constraints.* Prentice Hall, 2002

  Recommended reading: Chapters 1–3, 6 & 8

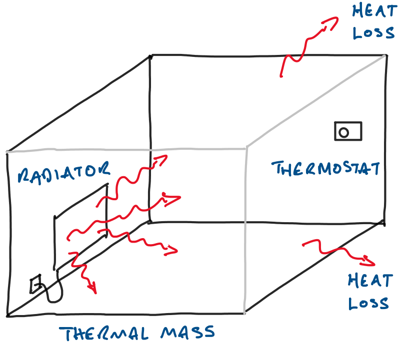
- J.B. Rawlings and D.Q. Mayne, *Model Predictive Control: Theory and Design.* Nob Hill Publishing, 2009

- B. Kouvaritakis and M. Cannon, *Model Predictive Control: Classical, Robust and Stochastic*, Springer 2015

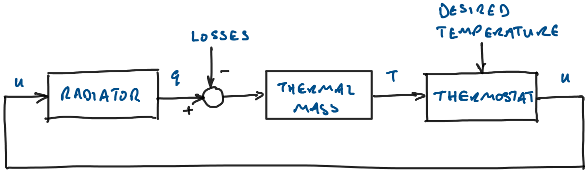
  Recommended reading: Chapters 1, 2 & 3

Motivating example: switching control

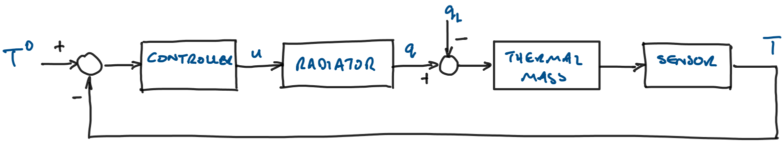
How does a thermostat regulate room temperature?



Closed loop control system:



Motivating example: switching control



System model:

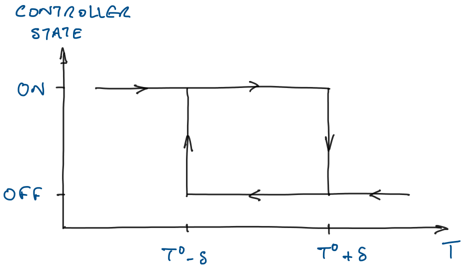
$$C\frac{dT}{dt} = q - q_L$$

$$q_L = \beta T$$

$$q = \alpha u$$

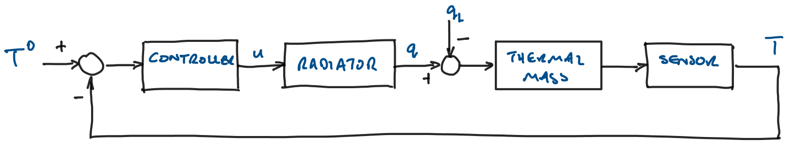
$$u = \begin{cases} U & \text{if ON} \\ 0 & \text{if OFF} \end{cases}$$

Switching controller:



- ⋆ Single controller parameter: hysteresis band δ
- ⋆ Accurate models aren't needed to regulate T to $[T^0 - \delta, T^0 + \delta]$

Motivating example: switching control



System model:

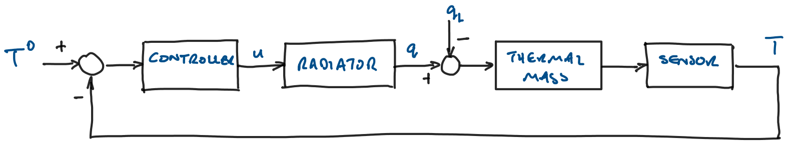
$$C\frac{dT}{dt} = q - q_L$$
$$q_L = \beta T$$
$$q = \alpha u$$
$$u = \begin{cases} U & \text{if ON} \\ 0 & \text{if OFF} \end{cases}$$

Closed loop response:

$$T(t) = T_{ss} + (T(0) - T_{ss})e^{-t/\tau}$$
$$T_{ss} = \begin{cases} \alpha U/\beta & \text{if ON} \\ 0 & \text{if OFF} \end{cases}$$
$$\tau = \frac{C}{\beta}$$

- ⋆ Single controller parameter: hysteresis band δ
- ⋆ Accurate models aren't needed to regulate T to $[T^0 - \delta, T^0 + \delta]$

Motivating example: switching control



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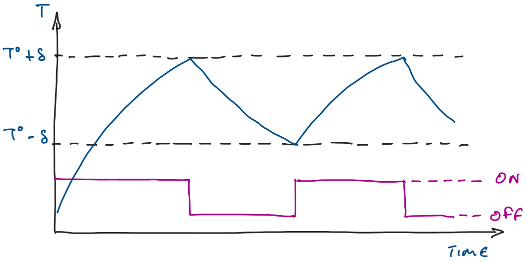
Closed loop response:



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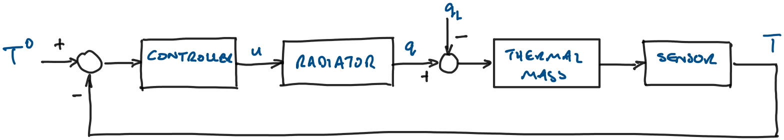
Motivating example: proportional control (P)



System model:

$$C\frac{dT}{dt} = q - q_L$$
$$q_L = \beta T$$
$$q = \alpha u$$
$$u = K(T^0 - T)$$

Closed loop response:

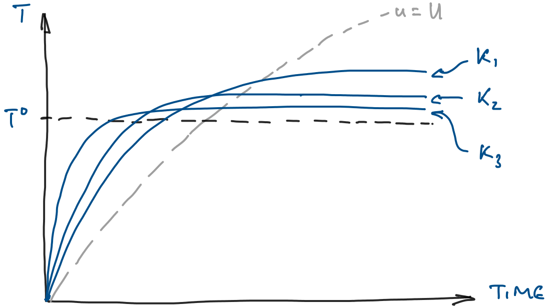
$$T(t) = T_{ss} + (T(0) - T_{ss})e^{-t/\tau}$$
$$T_{ss} = \frac{\alpha K}{\alpha K + \beta} T^0$$
$$\tau = \frac{C}{\alpha K + \beta}$$

- Controller parameter: gain K
- $T_{ss} \to T^0$ and $\tau \to 0$ as $K \to \infty$ independent of parameters C, α, β

Motivating example: proportional control (P)

Controller: $u = K(T^0 - T)$

Effect of increasing gain (ideal case), $K_1 < K_2 < K_3$:



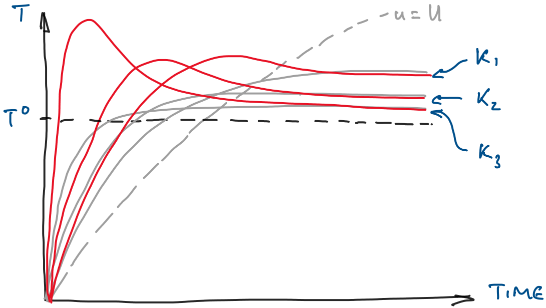
High gain K is often de-stabilizing because of:

- ⋆ nonlinearity, e.g. actuator saturation: $u = \min\left\{\bar{u}, \max\{K(T^0 - T), 0\}\right\}$
- ⋆ additional dynamics, e.g. sensor and actuator time-delay or lag

Motivating example: proportional control (P)

Controller: $u = K(T^0 - T)$

Actual effect of increasing gain:



High gain K is often de-stabilizing because of:

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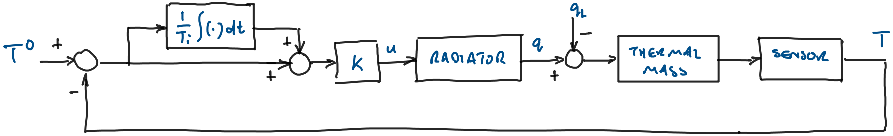
- ⋆ additional dynamics, e.g. sensor and actuator time-delay or lag

Motivating example: proportional + integral control (PI)

Control signal proportional to tracking error and integral of tracking error:

$$u = K(T^0 - T) + \frac{K}{T_i} \int^t (T^0 - T)\, dt$$



⋆ If closed loop system is stable
then $T^0 - T(t) \to 0$ as $t \to \infty$, i.e. no steady state error
(assuming T^0 = constant)

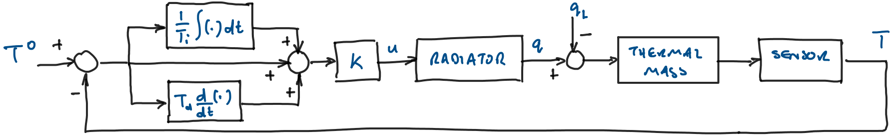
⋆ Controller has no knowledge of model parameters
but increasing gain (K/T_i) generally degrades transient performance
(overshoot and oscillations)

⋆ Two controller parameters K, T_i to be tuned/optimized

Motivating example: PID control

Include the rate of change of tracking error:

$$u = K(T^0 - T) + \frac{K}{T_i}\int^t (T^0 - T)\,dt + KT_d\frac{d}{dt}(T^0 - T)$$

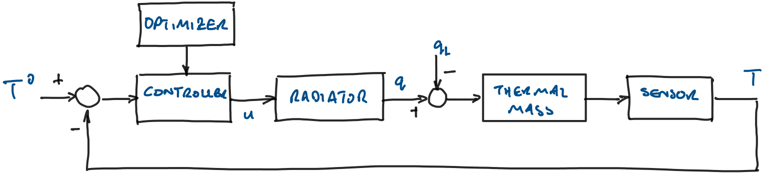


- ⋆ The derivative term provides anticipation of future error ("feedforward")
- ⋆ Three PID gains K, T_i, T_d need tuning, either using a system model or heuristic rules (e.g. Ziegler-Nichols)
- ⋆ PID tuning is difficult with nonlinear dynamics and constraints
- ⋆ Not obvious how to configure feedback loops for MIMO problems

Controller optimization

Can we optimize controller parameters for a given performance criterion?

e.g. mean square error: $\min_{K,T_i,T_d} \int_0^\infty \mathbb{E}\{(T^0 - T)^2 + \rho u^2\}\, dt$



⋆ Optimization of linear controller gains (e.g. K, T_i, T_d) is generally nonconvex

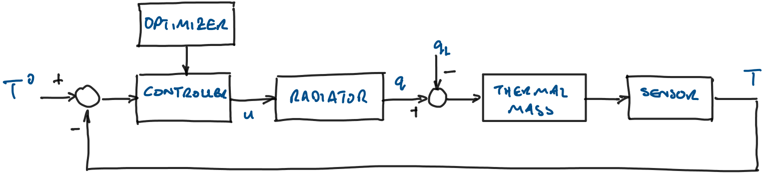
⋆ It's more common to optimize over control signals (e.g. LQG control)

$$\min_u \int_0^\infty \mathbb{E}\{(T^0 - T)^2 + \rho\, u^2\}\, dt$$

Unconstrained linear system $\implies$ solution is linear state feedback
but no closed-form solution in almost all other cases

Model predictive control

MPC optimizes predicted performance numerically over future control and state trajectories



- ⋆ The optimization is generally easier than optimizing feedback gains
  (e.g. convex for linear systems with linear state and input constraints)
- ⋆ Single-shot solution is an open loop control signal
  MPC updates it by repeating the optimization periodically online
- ⋆ This results in a feedback controller,
  providing robustness to model and measurement uncertainty
  and compensating for using finite numbers of optimization variables

Model predictive control

1. Prediction using a dynamic model & constraints
2. Online optimization
3. Receding horizon implementation

1. Prediction

- ⋆ Plant model: $x_{k+1} = f(x_k, u_k)$
- ⋆ Simulate forward in time (over a prediction horizon of N steps)

predicted input sequence $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$

predicted state sequence $\mathbf{x}_k = \begin{bmatrix} x_{0|k} \\ x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix}$

Notation: $(u_{i|k}, x_{i|k})$ = predicted i steps ahead | evaluated at time k

$$x_{0|k} = x_k$$

Overview of MPC

2. Optimization

   ⋆ Performance cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N} \ell_i(x_{i|k}, u_{i|k})$

   $\ell_i(x, u)$: stage cost

   ⋆ Optimize numerically to determine the optimal input sequence:

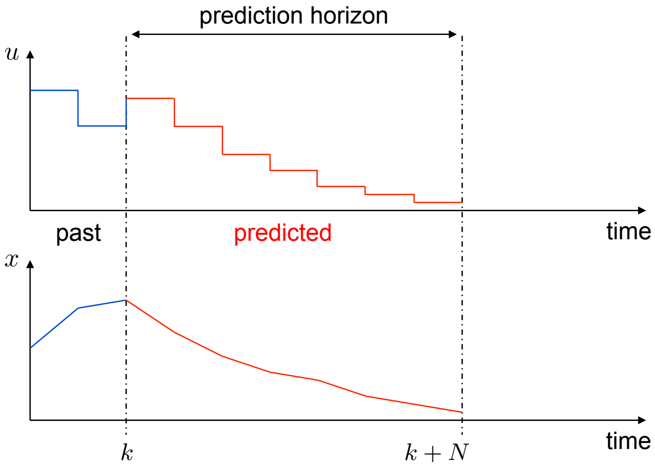
$$\begin{aligned} \mathbf{u}_k^* &= \arg\min_{\mathbf{u}_k} J(x_k, \mathbf{u}_k) \\ &= \big(u_{0|k}^*(x_k), \ldots, u_{N-1|k}^*(x_k)\big) \end{aligned}$$

3. Implementation

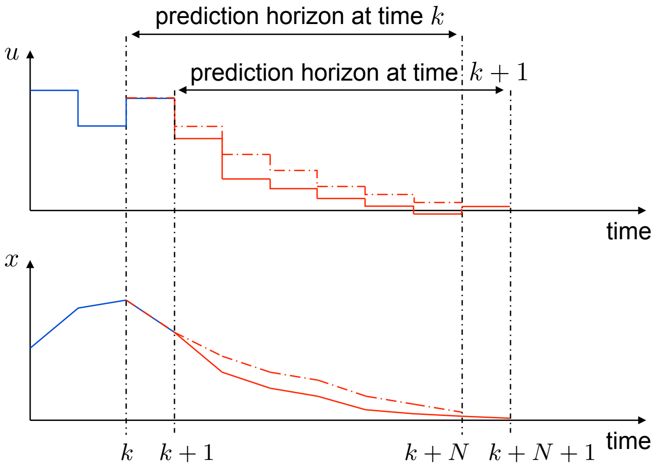
   ⋆ Use first element of $\mathbf{u}_k^*$ $\Longrightarrow$ MPC law: $u_k = u_{0|k}^*(x_k)$

   ⋆ Repeat optimization at each sampling instant $k = 0, 1, \ldots$

Overview of MPC



Overview of MPC



Example

Plant model:

$$x_{k+1} = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} -1 & 1 \end{bmatrix} x_k$$

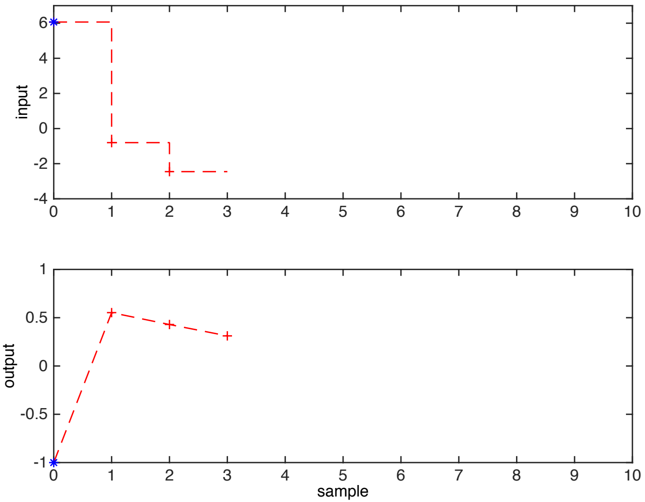
Cost:

$$\sum_{i=0}^{N-1} (y_{i|k}^2 + u_{i|k}^2) + y_{N|k}^2$$

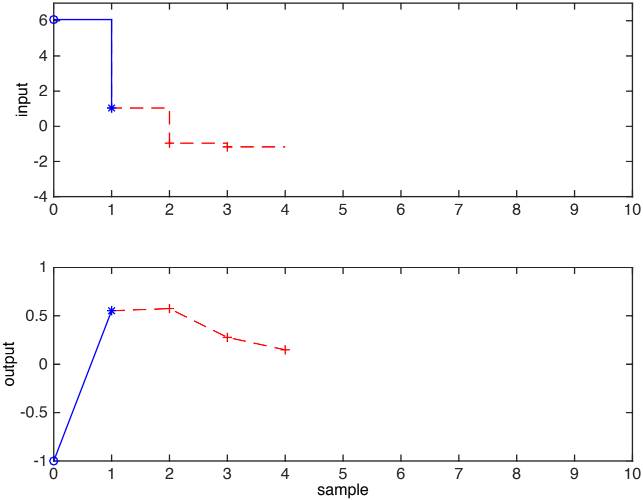
Prediction horizon: $N = 3$

Predicted input and state sequences: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ u_{2|k} \end{bmatrix}, \ \mathbf{x}_k = \begin{bmatrix} x_{0|k} \\ x_{1|k} \\ x_{2|k} \\ x_{3|k} \end{bmatrix}$

Example

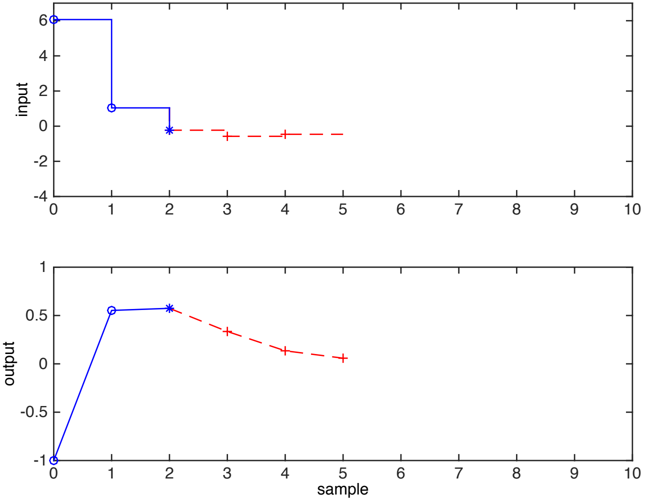


Example

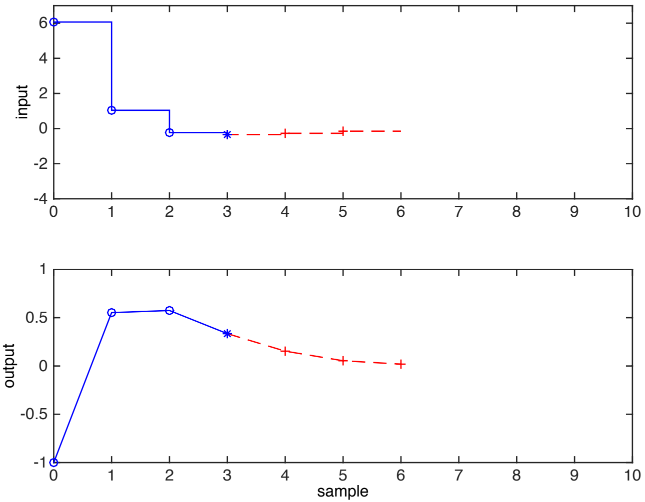


input
6
4
2
0
-2
-4
0 1 2 3 4 5 6 7 8 9 10
output
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10
sample

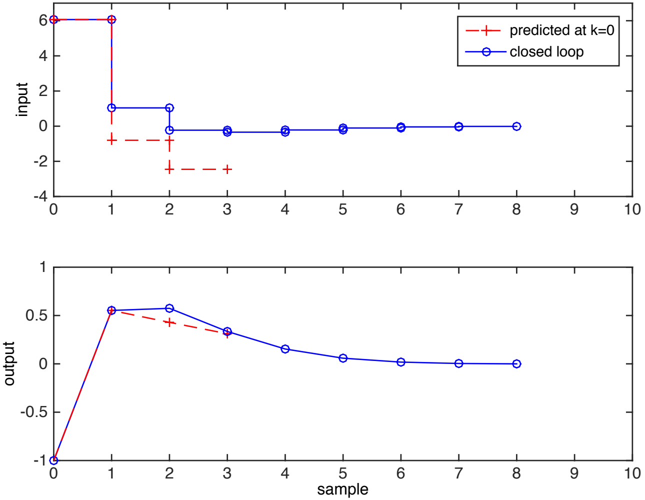
Example



Example



Example



predicted at k=0
closed loop
input
6
4
2
0
-2
-4
0 1 2 3 4 5 6 7 8 9 10
output
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10
sample

Model predictive control

Advantages

- Flexible plant model
  - multivariable
  - linear or nonlinear
  - deterministic, stochastic or fuzzy
- Handles constraints on control inputs and states
  - actuator limits
  - safety, environmental and economic constraints
- Approximately optimal control

Disadvantages

- Requires online optimization
  - quadratic programming (QP) problem for linear-quadratic problems
  - high computational requirement for nonlinear systems

MPC development

Control strategy reinvented several times

LQG optimal control	1950's
industrial process control	1980's
constrained nonlinear MPC	1990's
robust MPC	2000's
stochastic MPC	2010's

Current research challenges:

- high sample rates, long prediction horizons, uncertain & nonlinear models
- embedded optimization & sparse solvers
- adaptive and stochastic MPC

Prediction model

Linear plant model: $x_{k+1} = Ax_k + Bu_k$

- ▷ Predicted $\mathbf{x}_k$ depends linearly on $\mathbf{u}_k$ [details in Lecture 2]

- ▷ Therefore LQ cost is quadratic in $\mathbf{u}_k$ $\quad \mathbf{u}_k^\top H \mathbf{u}_k + 2f^\top \mathbf{u}_k + g(x_k)$

  and constraints are linear $\quad A_c \mathbf{u}_k \leq b(x_k)$

- ▷ Online optimization:

$$\min_{\mathbf{u}} \quad \mathbf{u}^\top H \mathbf{u} + 2f^\top \mathbf{u} \quad \text{s.t.} \quad A_c \mathbf{u} \leq b_c$$

This is a convex Quadratic Program (QP),
which is reliably and efficiently solvable

Prediction model

Nonlinear plant model: $x_{k+1} = f(x_k, u_k)$

- ▷ Predicted $\mathbf{x}_k$ depends nonlinearly on $\mathbf{u}_k$

- ▷ In general the cost is nonconvex in $\mathbf{u}_k$: $J(x_k, \mathbf{u}_k)$

  and the constraints are nonconvex: $g_c(x_k, \mathbf{u}_k) \le 0$

- ▷ Online optimization:

$$\min_{\mathbf{u}} \quad J(x_k, \mathbf{u}) \quad \text{s.t.} \quad g_c(x_k, \mathbf{u}) \le 0$$

  - may be nonconvex
  - may have local minima
  - may not be solvable efficiently or reliably

Prediction model

Discrete time prediction model

- Predictions optimized periodically at $t = 0, T, 2T, \ldots$
- Usually $T = T_s$ = sampling interval of model
- But $T = nT_s$ for any integer $n \geq 1$ is possible, (e.g. if $T_s <$ time needed for online optimization)

Prediction model

Continuous time prediction model

- Predicted $u(t)$ need not be piecewise constant,

  e.g. continuous, piecewise linear $u(t)$
  or $u(t) =$ polynomial in t (piecewise quadratic, cubic etc)

- Continuous time prediction models can be solved online

- This course: discrete-time model and $T = T_s$ assumed

Constraints

Classify state and input constraints as either hard or soft

- Hard constraints must be satisfied at all times,
  if this is not possible, then the problem is infeasible
- Soft constraints can be violated to avoid infeasibility
- Strategies for handling soft constraints:
  - impose (hard) constraints on the probability of violating each soft constraint
  - or remove active constraints until the problem becomes feasible

Constraints

Typical methods for handling input constraints:

(a) Saturate the unconstrained control law
(ignore constraints in controller design)

(b) De-tune the unconstrained control law
by increasing the penalty on u in the performance objective

(c) Use an anti-windup strategy to limit the state of a dynamic controller
(typically the integral term of a PI or PID controller)

(d) Use MPC with inequality-constrained optimization

Example: input constraints

(a) Effects of controller saturation, $\underline{u} \le u_k \le \overline{u}$

$$\text{unconstrained LQ optimal control:} \quad u^0(x) = K_{\text{LQ}}x$$

$$\text{saturated:} \quad u = \max\{\min\{u^0, \overline{u}\}, \underline{u}\}$$

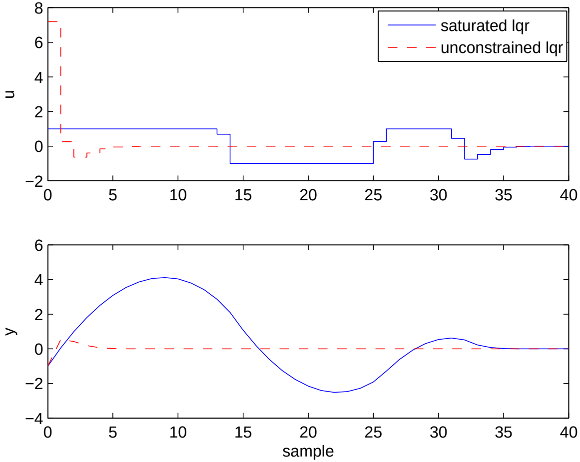
Input constraints:

$$\underline{u} \le u \le \overline{u}$$

$$\underline{u} = -1, \quad \overline{u} = 1$$

Controller saturation causes

- ⋆ poor performance
- ⋆ possible instability



Example: input constraints

(b) Effects of de-tuning the unconstrained optimal control law:

$$K_{\mathrm{LQ}} = \text{optimal gain for LQ cost} \sum_{k=0}^{\infty} (y_k^2 + \rho\, u_k^2)$$

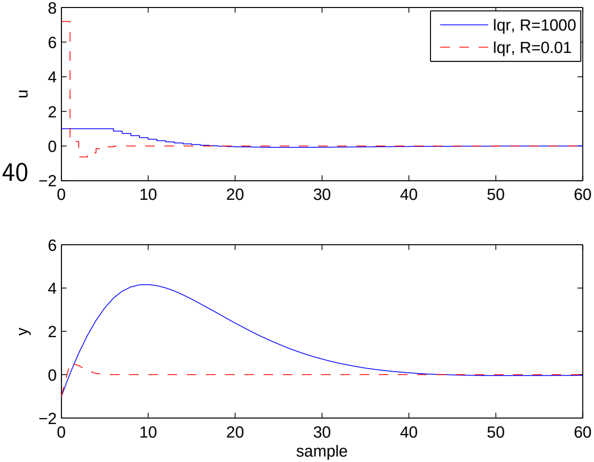
Increase ρ until $u = K_{\mathrm{LQ}} x$ satisfies constraints (locally)

Example

ρ increased from 10^{-2} to 10^{3}

settling time increased from 6 to 40

- ⋆ $y_k \to 0$ slowly
- ⋆ stability ensured
  (but here the response is
  slower than saturated LQR)



Example: input constraints

(c) Effects of Anti-windup:

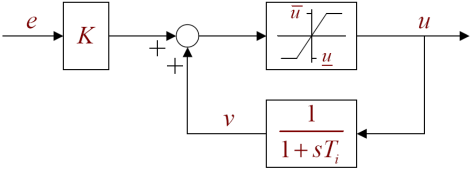
Anti-windup attempts to avoid instability while control input saturated

Many possible approaches, e.g. anti-windup PI controller:

$$u = \max\{\min\{(Ke + v), \overline{u}\}, \underline{u}\}$$

$$T_i \dot{v} + v = u$$

$$\Downarrow$$



$$\underline{u} \le u \le \overline{u} \quad \Longrightarrow \quad u = K\Big(e + \frac{1}{T_i}\int^t e\,dt\Big)$$

$u = \underline{u}$ or $\overline{u}$ $\quad \Longrightarrow \quad$ $v(t) \to \underline{u}$ or $\overline{u}$ exponentially

Heuristic strategy may not prevent instability

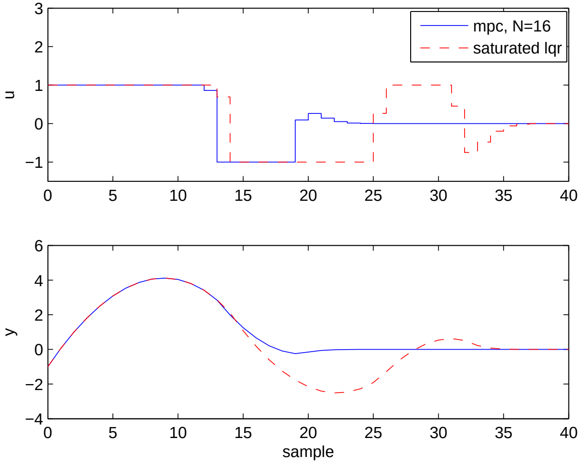
Example: input constraints

(d) Comparison with MPC (with prediction horizon $N = 16$)

Example

MPC vs saturated LQ
(both using the same cost):

- ⋆ settling time reduced to 20
- ⋆ stability is guaranteed



Summary

- Predict performance using plant model

  e.g. linear or nonlinear, discrete or continuous time

- Optimize future (open loop) control sequence

  computationally much easier than optimizing over feedback laws

- Implement first sample, then repeat optimization

  provides feedback to reduce effect of uncertainty

- Comparison of common methods of handling constraints:

  saturation, de-tuning, anti-windup, MPC

Lecture 2

Prediction and optimization

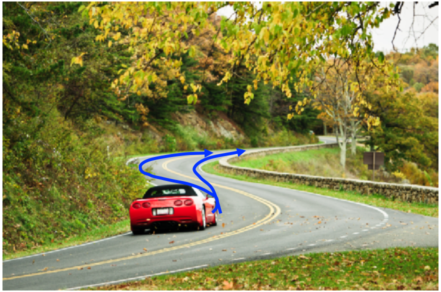
Prediction and optimization

- Input and state predictions
- Unconstrained finite horizon optimal control
- Infinite prediction horizons and connection with LQ optimal control
- Incorporating constraints
- Quadratic programming

Review of MPC strategy

At each sampling instant:

1. Use a model to predict system behaviour over a finite future horizon
2. Compute a control sequence by solving an online optimization problem
3. Apply the first element of optimal control sequence as control input



Advantages

- ⋆ flexible plant model
- ⋆ constraints taken into account
- ⋆ optimal performance

Disadvantage

- ⋆ online otimization required

Prediction equations

Linear time-invariant model: $x_{k+1} = Ax_k + Bu_k$

assume x_k is measured at time k

Predictions: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}, \quad \mathbf{x}_k = \begin{bmatrix} x_{0|k} \\ \vdots \\ x_{N|k} \end{bmatrix}$

Quadratic cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2$

$(\|x\|_Q^2 = x^\top Qx, \;\; \|u\|_R^2 = u^\top Ru$

$P =$ terminal weighting matrix)

Prediction equations

Linear time-invariant model: $x_{i+1|k} = Ax_{i|k} + Bu_{i|k}$

assume x_k is measured at time k

$$
\begin{aligned}
x_{0|k} &= x_k \\
x_{1|k} &= Ax_k + Bu_{0|k} \\
&\ \vdots \\
x_{N|k} &= A^N x_k + A^{N-1}Bu_{0|k} + A^{N-2}Bu_{1|k} + \cdots + Bu_{N-1|k}
\end{aligned}
$$

$$\Downarrow$$

$$\mathbf{x}_k = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k,$$

$$\mathcal{M} = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hline B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}$$

Prediction equations

Predicted cost:

$$J_k = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2$$

$$= \mathbf{x}_k^\top \mathbf{Q}\, \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R}\, \mathbf{u}_k \qquad \begin{cases} \mathbf{Q} = \mathrm{diag}\{Q, \ldots, Q, P\} \\ \mathbf{R} = \mathrm{diag}\{R, \ldots, R, R\} \end{cases}$$

$$\Downarrow$$

$$\boxed{J_k = \mathbf{u}_k^\top H \mathbf{u}_k + 2x_k^\top F^\top \mathbf{u}_k + x_k^\top G x_k}$$

where

$$\begin{aligned} H &= \mathcal{C}^\top \mathbf{Q}\, \mathcal{C} + \mathbf{R} && \leftarrow \ \mathbf{u} \times \mathbf{u} \text{ terms} \\ F &= \mathcal{C}^\top \mathbf{Q}\, \mathcal{M} && \leftarrow \ \mathbf{u} \times x \text{ terms} \\ G &= \mathcal{M}^\top \mathbf{Q}\, \mathcal{M} && \leftarrow \ x \times x \text{ terms} \end{aligned}$$

time-invariant model $\Longrightarrow$ H, F, G can be computed offline

Prediction equations

Predicted cost:

$$J_k = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2$$

$$= \mathbf{x}_k^\top \mathbf{Q}\, \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R}\, \mathbf{u}_k \qquad \begin{cases} \mathbf{Q} = \mathrm{diag}\{Q, \ldots, Q, P\} \\ \mathbf{R} = \mathrm{diag}\{R, \ldots, R, R\} \end{cases}$$

$$\Downarrow$$

$$\boxed{J_k = \mathbf{u}_k^\top H \mathbf{u}_k + 2 x_k^\top F^\top \mathbf{u}_k + x_k^\top G x_k}$$

where

$$H = \mathcal{C}^\top \mathbf{Q}\, \mathcal{C} + \mathbf{R} \quad \leftarrow \quad \mathbf{u} \times \mathbf{u} \text{ terms}$$

$$F = \mathcal{C}^\top \mathbf{Q}\, \mathcal{M} \quad \leftarrow \quad \mathbf{u} \times x \text{ terms}$$

$$G = \mathcal{M}^\top \mathbf{Q}\, \mathcal{M} \quad \leftarrow \quad x \times x \text{ terms}$$

time-invariant model $\Longrightarrow$ H, F, G can be computed offline

Prediction equations – example

Plant model: $x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

Prediction horizon $N = 4$:
$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.079 & 0 & 0 & 0 \\ 0.157 & 0 & 0 & 0 \\ 0.075 & 0.079 & 0 & 0 \\ 0.323 & 0.157 & 0 & 0 \\ 0.071 & 0.075 & 0.079 & 0 \\ 0.497 & 0.323 & 0.157 & 0 \\ 0.068 & 0.071 & 0.075 & 0.079 \end{bmatrix}$$

Cost matrices $Q = C^\top C$, $R = 0.01$, and $P = Q$:

$$H = \begin{bmatrix} 0.271 & 0.122 & 0.016 & -0.034 \\ \star & 0.086 & 0.014 & -0.020 \\ \star & \star & 0.023 & -0.007 \\ \star & \star & \star & 0.016 \end{bmatrix} \qquad F = \begin{bmatrix} 0.977 & 4.925 \\ 0.383 & 2.174 \\ 0.016 & 0.219 \\ -0.115 & -0.618 \end{bmatrix}$$

$$G = \begin{bmatrix} 7.589 & 22.78 \\ \star & 103.7 \end{bmatrix}$$

Prediction equations: LTV model

Linear time-varying model: $x_{k+1} = A_k x_k + B_k u_k$

assume x_k is measured at time k

Predictions:

$$
\begin{aligned}
x_{0|k} &= x_k \\
x_{1|k} &= A_k x_k + B_k u_{0|k} \\
x_{2|k} &= A_{k+1} A_k x_k + A_{k+1} B_k u_{0|k} + B_{k+1} u_{1|k} \\
&\ \vdots \\
x_{i|k} &= \prod_{j=i-1}^{0} A_{k+j} x_k + \mathcal{C}_i(k) \mathbf{u}_k, \qquad i = 0, \ldots, N \\
\mathcal{C}_i(k) &= \begin{bmatrix} \prod_{j=i-1}^{1} A_{k+j} B_k & \prod_{j=i-1}^{2} A_{k+j} B_{k+1} & \cdots & B_{k+i-1} & 0 & \cdots & 0 \end{bmatrix}
\end{aligned}
$$

- ⋆ $\prod_{j=i-1}^{0} A_{k+j} = A_{k+i-1} \cdots A_k$ for $i \geq 1$ and $\prod_{j=i-1}^{0} A_{k+j} = 0$ for $i = 0$
- ⋆ $H(k)$, $F(k)$, $G(k)$ depend on k and must be computed online

Unconstrained optimization

Minimize cost: $\mathbf{u}^* = \arg\min_{\mathbf{u}} J, \quad J = \mathbf{u}^\top H \mathbf{u} + 2x^\top F^\top \mathbf{u} + x^\top G x$

differentiate w.r.t. $\mathbf{u}$: $\nabla_{\mathbf{u}} J = 2H\mathbf{u} + 2Fx = 0$

$$\Downarrow$$

$$\mathbf{u} = -H^{-1}Fx$$

$= \mathbf{u}^*$ if H is positive definite i.e. if $H \succ 0$

Here $H = \mathcal{C}^\top \mathbf{Q} \mathcal{C} + \mathbf{R} \succ 0$ if: $\begin{cases} R \succ 0 \ \&\ Q, P \succeq 0 & \text{or} \\ R \succeq 0 \ \&\ Q, P \succ 0 \ \&\ \mathcal{C} \text{ is full-rank} \end{cases}$

$$\Updownarrow$$

(A, B) controllable

Receding horizon controller is linear state feedback:

$$u_k = -\begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} H^{-1} F x_k$$

is the closed loop response optimal? is it even stable?

Unconstrained optimization

Minimize cost: $\mathbf{u}^* = \arg\min_{\mathbf{u}} J, \quad J = \mathbf{u}^\top H \mathbf{u} + 2x^\top F^\top \mathbf{u} + x^\top G x$

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$$\Downarrow$$

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$$= \mathbf{u}^* \quad \text{if } H \text{ is positive definite i.e. if } H \succ 0$$

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$$\Updownarrow$$

(A, B) controllable

Receding horizon controller is linear state feedback:

$$u_k = -\begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} H^{-1} F x_k$$

is the closed loop response optimal? is it even stable?

Example

Model: A, B, C as before, cost: $J_k = \sum_{i=0}^{N-1} \left(y_{i|k}^2 + 0.01 u_{i|k}^2\right) + y_{N|k}^2$

- For $N = 4$: $\quad \mathbf{u}_k^* = -H^{-1} F x_k = \begin{bmatrix} -4.36 & -18.7 \\ 1.64 & 1.24 \\ 1.41 & 3.00 \\ 0.59 & 1.83 \end{bmatrix} x_k$

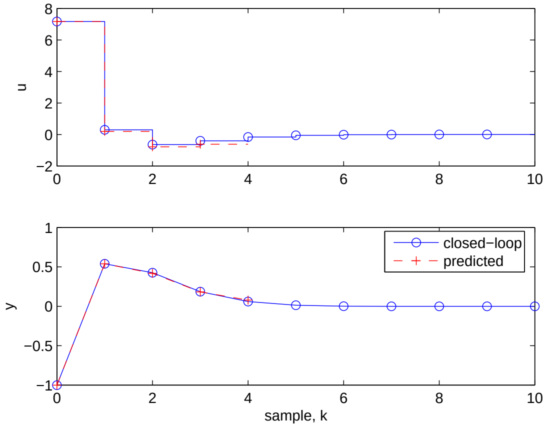
$$u_k = \begin{bmatrix} -4.36 & -18.7 \end{bmatrix} x_k$$

- For general N: $\quad u_k = L(N) x_k$

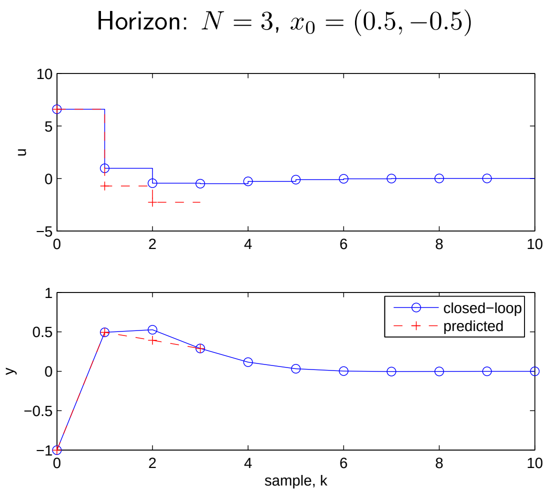
	$N = 4$	$N = 3$	$N = 2$	$N = 1$
$L(N)$	$[-4.36 \quad -18.69]$	$[-3.80 \quad -16.98]$	$[1.22 \quad -3.95]$	$[5.35 \quad 5.10]$
$\lambda\big(A + BL(N)\big)$	$0.29 \pm 0.17j$	$0.36 \pm 0.22j$	$1.36, 0.38$	$2.15, 0.30$
	stable	stable	unstable	unstable

Example

Horizon: $N = 4,\ x_0 = (0.5, -0.5)$

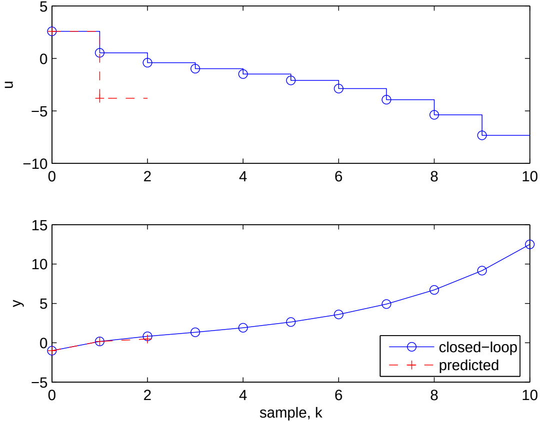


Example


Horizon: $N = 3$, $x_0 = (0.5, -0.5)$
u
10
5
0
−5
0
2
4
6
8
10
y
1
0.5
0
−0.5
−1
closed−loop
predicted
sample, k

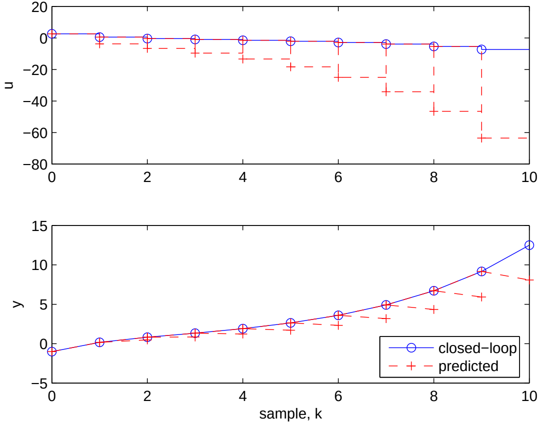
Example

Horizon: $N = 2,\ x_0 = (0.5, -0.5)$



Example

Horizon: $N = 2$, $x_0 = (0.5, -0.5)$



Observation: big differences exist between predicted and closed loop responses for small N

Receding horizon control

Why is this example unstable for $N \le 2$?

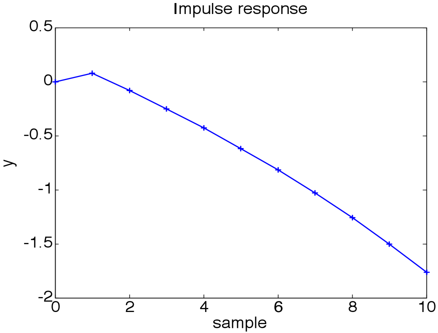
System is non-minimum phase

$$\Downarrow$$

impulse response changes sign

$$\Downarrow$$

therefore short horizon causes instability



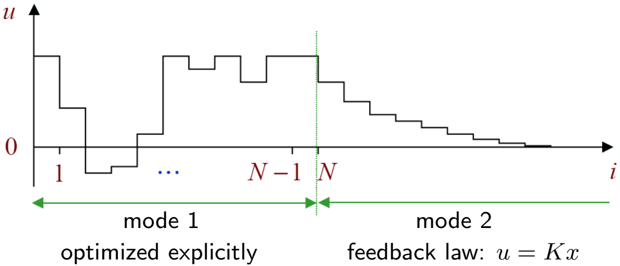
Solution:

- ⋆ use an infinite horizon cost
- ⋆ but keep a finite number of optimization variables in predictions

Dual mode predictions

An infinite prediction horizon is possible with dual mode predictions:

$$u_{i|k} = \begin{cases} \text{optimization variables} & i = 0, \ldots, N-1, \quad \text{mode 1} \\ Kx_{i|k} & i = N, N+1, \ldots \quad \text{mode 2} \end{cases}$$



Feedback gain K: stabilizing and determined offline

e.g. unconstrained LQ optimal for $\sum_{i=0}^{\infty}(\|x_i\|_Q^2 + \|u_i\|_R^2)$

Infinite horizon cost

If the predicted input sequence is

$$\{u_{0|k}, \dots, u_{N-1|k}, Kx_{N|k}, K\Phi x_{N|k}, \dots\}$$

then

$$\sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) + \|x_{N|k}\|_P^2$$

where

$$\boxed{P - (A+BK)^\top P(A+BK) = Q + K^\top RK}$$

Lyapunov matrix equation (discrete time)

- If $Q + K^\top RK \succ 0$, then the solution P is unique and $P \succ 0$
- Matlab: `P = dlyap(Phi',RHS);`
  `Phi = A+B*K; RHS = Q+K'*R*K;`
- P is equal to the steady state Riccati equation solution if K is LQ optimal

Infinite horizon cost

If the predicted input sequence is

$$\{u_{0|k}, \ldots, u_{N-1|k}, Kx_{N|k}, K\Phi x_{N|k}, \ldots\}$$

then

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where

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Lyapunov matrix equation (discrete time)

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```
P = dlyap(Phi',RHS);
Phi = A+B*K; RHS = Q+K'*R*K;
```

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Infinite horizon cost

Proof that the predicted cost over the mode 2 horizon is $\|x_{N|k}\|_P^2$:

Let $J^\infty(x) = \sum_{i=0}^{\infty}\big(\|x_i\|_Q^2 + \|u_i\|_R^2\big)$, with $u_i = Kx_i,\ x_{i+1} = \Phi x_i\ \forall i$

$$x_0 = x$$

- then $J^\infty(x) = \sum_{i=0}^{\infty}\big(x^\top \Phi^{i\top} Q \Phi^i x + x^\top K^\top \Phi^{i\top} R K \Phi^i x\big)$

$$= x^\top \underbrace{\left[\sum_{i=0}^{\infty} (\Phi^i)^\top (Q + K^\top R K)\Phi^i\right]}_{=P} x = \|x\|_P^2$$

- but $\Phi^\top P \Phi = \sum_{i=1}^{\infty} (\Phi^i)^\top (Q + K^\top R K)\Phi^i$

$$= P - (Q + K^\top R K)$$

so $P - \Phi^\top P \Phi = Q + K^\top R K$

Infinite horizon cost

Proof that the predicted cost over the mode 2 horizon is $\|x_{N|k}\|_P^2$:

Let $J^\infty(x) = \sum_{i=0}^{\infty}\big(\|x_i\|_Q^2 + \|u_i\|_R^2\big)$, with $u_i = Kx_i,\ x_{i+1} = \Phi x_i\ \forall i$
$x_0 = x$

- then $$J^\infty(x) = \sum_{i=0}^{\infty}\big(x^\top {\Phi^i}^\top Q \Phi^i x + x^\top K^\top {\Phi^i}^\top R K \Phi^i x\big)$$
$$= x^\top \underbrace{\Big[\sum_{i=0}^{\infty}(\Phi^i)^\top (Q + K^\top R K)\Phi^i\Big]}_{=P} x = \|x\|_P^2$$
- but $$\Phi^\top P \Phi = \sum_{i=1}^{\infty}(\Phi^i)^\top (Q + K^\top R K)\Phi^i$$
$$= P - (Q + K^\top R K)$$

so $$P - \Phi^\top P \Phi = Q + K^\top R K$$

Connection with LQ optimal control

Let
$$J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) + \|x_{N|k}\|_P^2$$
$$P - (A+BK)^\top P(A+BK) = Q + K^\top RK, \qquad K = \text{LQ optimal}$$

Then the solution of the unconstrained optimization satisfies

$$\boxed{u^*_{0|k} = Kx_k \text{ where } \mathbf{u}^*_k = \arg\min_{\mathbf{u}} J(x_k, \mathbf{u}) = (u^*_{0|k}, \ldots, u^*_{N-1|k})}$$

since

$$\{u_{0|k}, u_{1,k}, \ldots\} \text{ is optimal iff } \begin{cases} \mathbf{u}_k = \{u_{0|k}, \ldots, u_{N-1|k}\} \text{ is optimal} \\ \text{and } \{u_{N|k}, u_{N+1|k}, \ldots\} \text{ is optimal} \end{cases}$$

Connection with LQ optimal control – example

- Model parameters (A, B, C) as before
  LQ optimal gain for $Q = C^\top C$, $R = 0.01$: $K = \begin{bmatrix} -4.36 & -18.74 \end{bmatrix}$
  Lyapunov equation solution: $P = \begin{bmatrix} 3.92 & 4.83 \\ & 13.86 \end{bmatrix}$

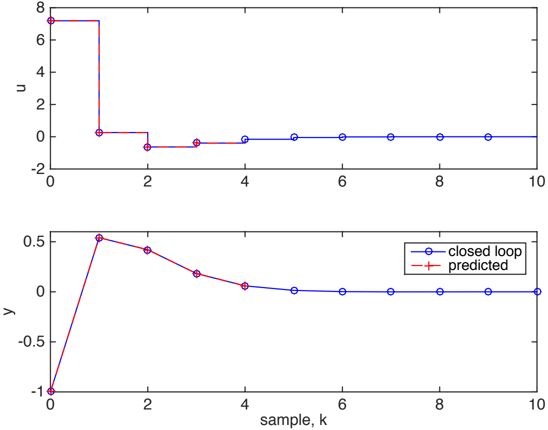
- Cost matrices for $N = 4$:

$$H = \begin{bmatrix} 1.44 & 0.98 & 0.59 & 0.26 \\ \star & 0.72 & 0.44 & 0.20 \\ \star & \star & 0.30 & 0.14 \\ \star & \star & \star & 0.096 \end{bmatrix} \quad F = \begin{bmatrix} 3.67 & 23.9 \\ 2.37 & 16.2 \\ 1.36 & 9.50 \\ 0.556 & 4.18 \end{bmatrix} \quad G = \begin{bmatrix} 13.8 & 66.7 \\ \star & 413 \end{bmatrix}$$

- Predictive control law: $u_k = -\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} H^{-1} F x_k$
  $= \begin{bmatrix} -4.35 & -18.74 \end{bmatrix} x_k$

Connection with LQ optimal control – example

- Response for $N = 4$, $x_0 = (0.5, -0.5)$



$$\left.\begin{array}{l}\text{Infinite horizon cost}\\ \text{no constraints}\end{array}\right\} \implies \text{identical predicted and closed loop responses}$$

Dual mode predictions

Pre-stabilize predictions to provide better numerical stability:

- Control inputs

$$\begin{array}{lll} \text{mode 1} & u_{i|k} = Kx_{i|k} + c_{i|k}, & i = 0, 1, \ldots, N-1 \\ \text{mode 2} & u_{i|k} = Kx_{i|k}, & i = N, N+1, \ldots \end{array}$$

- States

$$\begin{array}{lll} \text{mode 1} & x_{i+1|k} = \Phi x_{i|k} + Bc_{i|k}, & i = 0, 1, \ldots, N-1 \\ \text{mode 2} & x_{i+1|k} = \Phi x_{i|k}, & i = N, N+1, \ldots \end{array}$$

where $(c_{0|k}, \ldots, c_{N-1|k})$ are optimization variables

Dual mode predictions

Pre-stabilize predictions to provide better numerical stability:

- Vectorized form: $\mathbf{x}_k = \mathcal{M} x_k + \mathcal{C} \mathbf{c}_k$

$$\mathbf{x}_k := \begin{bmatrix} x_{0|k} \\ \vdots \\ x_{N|k} \end{bmatrix}, \quad \mathbf{c}_k := \begin{bmatrix} c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} I \\ \Phi \\ \Phi^2 \\ \vdots \\ \Phi^N \end{bmatrix}, \quad \mathcal{C} = \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \hline B & & & \\ \Phi B & B & & \\ \vdots & \vdots & \ddots & \\ \Phi^{N-1} B & \Phi^{N-2} B & \cdots & B \end{array}\right]$$

- Cost: $J\big(x_k, (u_{0|k}, \ldots, u_{N-1|k})\big) = \mathcal{J}(x_k, \mathbf{c}_k)$

Input and state constraints

Infinite horizon unconstrained MPC = LQ optimal control

but MPC can also handle constraints

Consider constraints applied to mode 1 predictions:

- ⋆ input constraints: $\underline{u} \le u_{i|k} \le \overline{u}, \quad i = 0, \ldots, N-1$

$$\iff \quad \begin{bmatrix} I \\ -I \end{bmatrix} \mathbf{u}_k \le \begin{bmatrix} \overline{\mathbf{u}} \\ -\underline{\mathbf{u}} \end{bmatrix} \qquad \text{where} \qquad \begin{aligned} \overline{\mathbf{u}} &= \begin{bmatrix} \overline{u}^\top & \cdots & \overline{u}^\top \end{bmatrix}^\top \\ \underline{\mathbf{u}} &= \begin{bmatrix} \underline{u}^\top & \cdots & \underline{u}^\top \end{bmatrix}^\top \end{aligned}$$

- ⋆ state constraints: $\underline{x} \le x_{i|k} \le \overline{x}, \quad i = 1, \ldots, N$

$$\iff \quad \begin{bmatrix} \mathcal{C}_i \\ -\mathcal{C}_i \end{bmatrix} \mathbf{u}_k \le \begin{bmatrix} \overline{x} \\ -\underline{x} \end{bmatrix} + \begin{bmatrix} -A^i \\ A^i \end{bmatrix} x_k, \quad i = 1, \ldots, N$$

Input and state constraints

Constraints on mode 1 predictions can be expressed

$$A_c \mathbf{u}_k \leq b_c + B_c x_k$$

where A_c, B_c, b_c can be computed offline since model is time-invariant

The online optimization is a quadratic program (QP):

$$\begin{aligned} \underset{\mathbf{u}}{\text{minimize}} \quad & \mathbf{u}^\top H \mathbf{u} + 2x_k^\top F^\top \mathbf{u} \\ \text{subject to} \quad & A_c \mathbf{u} \leq b_c + B_c x_k \end{aligned}$$

which is a convex optimization problem with a unique solution if

$$H = \mathcal{C}^\top \mathbf{Q} \mathcal{C} + \mathbf{R} \ \text{ is positive definite}$$

QP solvers: (a) Active set

Consider the QP:
$$\mathbf{u}^* = \arg\min_{\mathbf{u}} \quad \mathbf{u}^\top H \mathbf{u} + 2f^\top \mathbf{u}$$
$$\text{subject to} \quad A\mathbf{u} \le b$$

and let $(A_i, b_i) = i$th row/element of (A, b)

▷ Individual constraints are active or inactive

active	inactive
$A_i\mathbf{u}^* = b_i, \ \ \forall i \in \mathcal{I}$	$A_i\mathbf{u}^* \le b_i, \ \ \forall i \notin \mathcal{I}$
b_i affects solution	b_i does not affect solution

▷ Equality constraint problem:
$$\mathbf{u}^* = \arg\min_{\mathbf{u}} \quad \mathbf{u}^\top H \mathbf{u} + 2f^\top \mathbf{u}$$
$$\text{subject to} \quad A_i\mathbf{u} = b_i, \ \ \forall i \in \mathcal{I}$$

▷ Solve QP by searching for $\mathcal{I}$

- ⋆ one equality constraint problem solved at each iteration
- ⋆ optimality conditions (KKT conditions) identify solution

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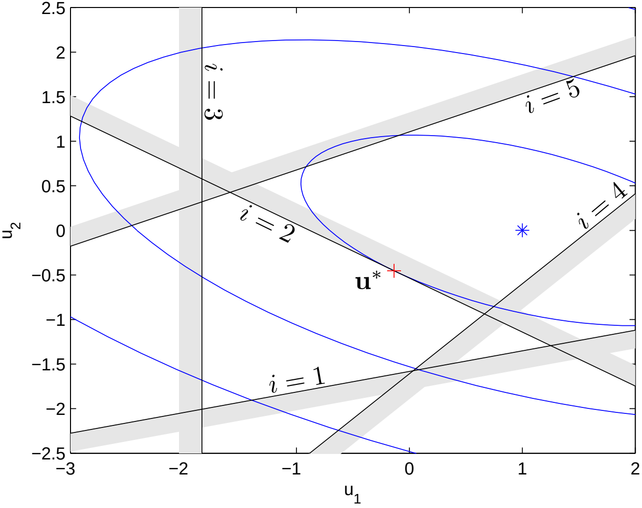
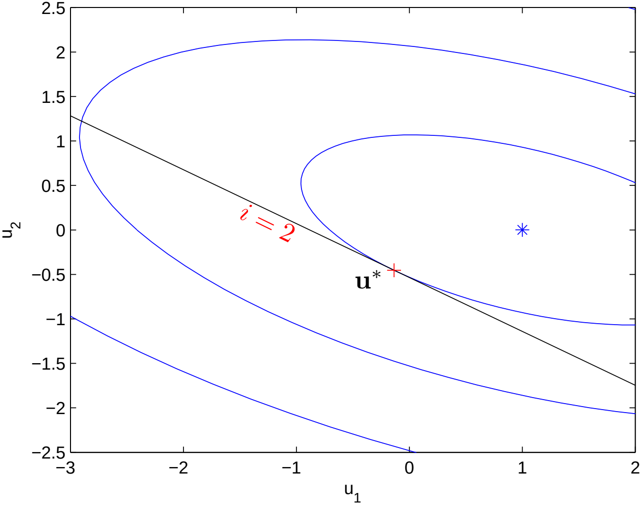
Active constraints – example



A QP problem with 5 inequality constraints

active set at solution: $\mathcal{I} = \{2\}$

Active constraints – example



An equivalent equality constraint problem

QP solvers: (a) Active set

- Computation:

  $O(N^3 n_u^3)$ additions/multiplications per iteration (conservative estimate)

  upper bound on number of iterations is exponential in problem size

- At each iteration choose trial active set using: cost gradient
  Lagrange multipliers (constraint sensitivities)

  The number of iterations needed is often small in practice

- In MPC $\mathbf{u}_k^* = \mathbf{u}^*(x_k)$ and $\mathcal{I}_k = \mathcal{I}(x_k)$

  hence initialize solver at time k using the solution computed at $k-1$

QP solvers: (b) Interior point

- Solve an unconstrained problem at each iteration:

$$\mathbf{u}(\mu) = \min_{\mathbf{u}} \mu\big(\mathbf{u}^\top H \mathbf{u} + 2 f^\top \mathbf{u}\big) + \phi(\mathbf{u})$$

  where

$$\phi(\mathbf{u}) = \text{barrier function} \quad (\phi \to \infty \text{ at constraints})$$
$$\mathbf{u} \to \mathbf{u}^* \text{ as } \mu \to \infty$$

  Increase μ until $\phi(\mathbf{u}^*) > 1/\epsilon$ ($\epsilon =$ user-defined tolerance)

- # arithmetic operations per iteration is constant, e.g. $O(N^3 n_u^3)$
  # iterations for given ϵ is polynomial in problem size

$$\Downarrow$$

  Computational advantages for large-scale problems
  e.g. # variables $> 10^2$, # constraints $> 10^3$

- No general method for initializing at solution estimate

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$$\Downarrow$$

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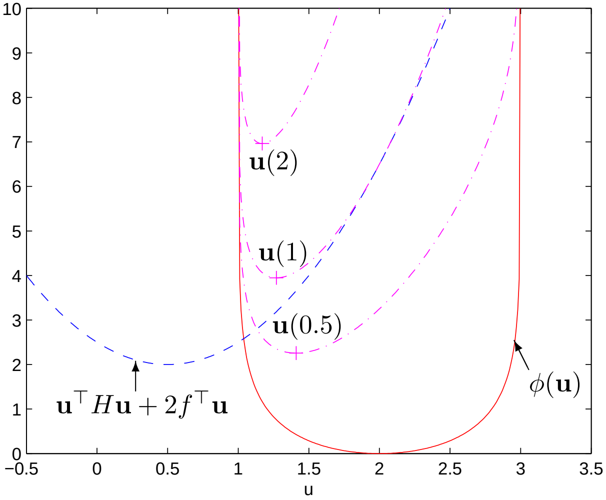
- No general method for initializing at solution estimate

Interior point method – example



$\mathbf{u}(\mu) \rightarrow \mathbf{u}^* = 1$ as $\mu \rightarrow \infty$

but $\min_{\mathbf{u}} \mu\big(\mathbf{u}^\top H\mathbf{u} + 2f^\top \mathbf{u}\big) + \phi(\mathbf{u})$ becomes ill-conditioned as $\mu \rightarrow \infty$

QP solvers: (c) Multiparametric

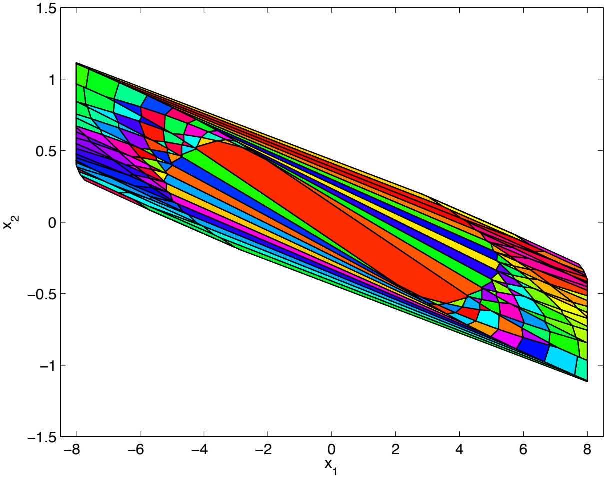
Let
$$\begin{aligned} \mathbf{u}^*(x) = \ & \underset{\mathbf{u}}{\arg\min} && \mathbf{u}^\top H \mathbf{u} + 2x^\top F^\top \mathbf{u} \\ & \text{subject to} && A\mathbf{u} \le b + Bx \end{aligned}$$

then:

- ⋆ $\mathbf{u}^*$ is a continous function of x
- ⋆ $\mathbf{u}^*(x) = K_j x + k_j$ for all x in a polytopic set $\mathcal{X}_j$

- ▷ In principle each K_j, k_j and $\mathcal{X}_j$ can be determined offline
- ▷ Large number of sets $\mathcal{X}_j$ (combinatorial in problem size) so online determination of j such that $x_k \in \mathcal{X}_j$ is difficult

Multiparametric QP – example



Model: (A, B, C) as before,

cost: $Q = C^\top C$, $R = 1$, horizon: $N = 10$

constraints: $-1 \le u \le 1$, $-\mathbf{1} \le x/8 \le \mathbf{1}$

Summary

▷ Predicted control inputs: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$

and states: $\mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix} = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k$

▷ Predicted cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) + \|x_{N|k}\|_P^2$

$$= \mathbf{u}_k^\top H \mathbf{u}_k + 2x_k^\top F^\top \mathbf{u}_k + x_k^\top G x_k$$

▷ Online optimization subject to linear state and input constraints is a QP:

$$\begin{aligned} \underset{\mathbf{u}}{\text{minimize}} \quad & \mathbf{u}^\top H \mathbf{u} + 2x_k^\top F^\top \mathbf{u} \\ \text{subject to} \quad & A_c \mathbf{u} \leq b_c + B_c x_k \end{aligned}$$

Lecture 3

Closed loop properties of MPC

Closed loop properties of MPC

- Review: infinite horizon cost
- Infinite horizon predictive control with constraints
- Closed loop stability
- Constraint-checking horizon
- Connection with constrained optimal control

Review: infinite horizon cost

Short prediction horizons cause poor performance and instability, so

- ⋆ use an infinite horizon cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{\infty} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \big)$

- ⋆ keep optimization finite-dimensional by using dual mode predictions:

$$u_{i|k} = \begin{cases} \text{optimization variables} & i = 0, \dots, N-1, \quad \text{mode 1} \\ K x_{i|k} & i = N, N+1, \dots \quad \text{mode 2} \end{cases}$$

mode 1: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$ $\qquad$ $\mathbf{u}_k$ optimized online

mode 2: $u_{i|k} = K x_{i|k}$ $\qquad$ K chosen offline

Review: infinite horizon cost

- Cost for mode 2: $\displaystyle\sum_{i=N}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) = \|x_{N|k}\|_P^2$

  P is the solution of the Lyapunov equation

$$\boxed{P - (A + BK)^\top P(A + BK) = Q + K^\top RK}$$

- Infinite horizon cost:

$$\begin{aligned} J(x_k, \mathbf{u}_k) &= \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) + \|x_{N|k}\|_P^2 \\ &= \mathbf{u}_k^\top H \mathbf{u}_k + 2x_k^\top F^\top \mathbf{u}_k + x_k^\top G x_k \end{aligned}$$

Review: MPC online optimization

- Unconstrained optimization: $\nabla_{\mathbf{u}} J(x, \mathbf{u}^*) = 2H\mathbf{u}^* + 2Fx = 0$, so

$$\mathbf{u}^*(x) = -H^{-1}Fx$$

$\implies$ linear controller: $u_k = K_{\mathsf{MPC}} x_k$

$$K_{\mathsf{MPC}} = \text{LQ-optimal if } K = \text{LQ-optimal (in mode 2)}$$

- Constrained optimization:

$$\begin{aligned} \mathbf{u}^*(x) = \arg\min_{\mathbf{u}} \quad & \mathbf{u}^\top H \mathbf{u} + 2x^\top F^\top \mathbf{u} \\ \text{subject to} \quad & A_c \mathbf{u} \leq b_c + B_c x \end{aligned}$$

$\implies$ nonlinear controller: $u_k = K_{\mathsf{MPC}}(x_k)$

Constrained MPC – example

- Plant model: $x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$

  $$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

  Constraints: $-1 \le u_k \le 1$

- MPC optimization (constraints applied only to mode 1 predictions):

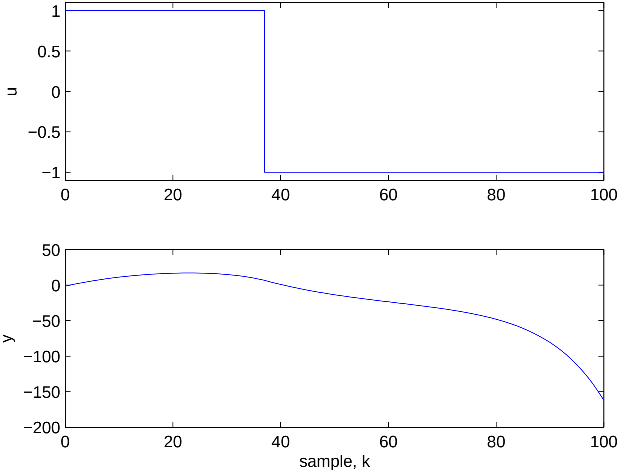
$$\begin{aligned} \underset{\mathbf{u}}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2 \\ \text{subject to} \quad & -1 \le u_{i|k} \le 1, \quad i = 0, \ldots, N-1 \end{aligned}$$

$$Q = C^\top C, \; R = 0.01, \; N = 2$$

. . . performance? stability?

Constrained MPC – example

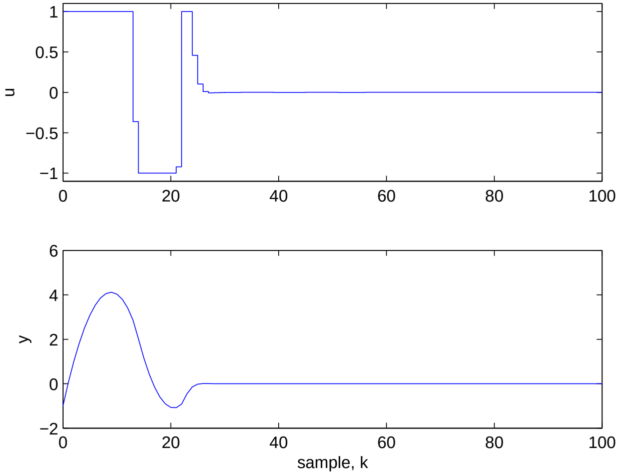
Closed loop response for $x_0 = (0.8, -0.8)$



unstable

Constrained MPC – example

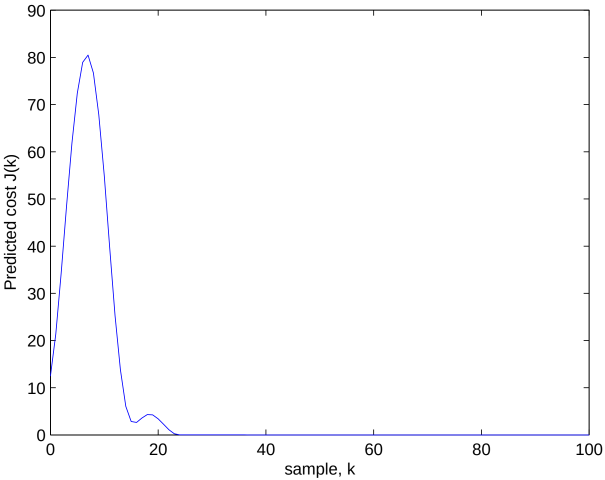
Closed loop response for $x_0 = (0.5, -0.5)$



stable, but . . .

Constrained MPC – example

Optimal predicted cost $x_0 = (0.5, -0.5)$



…increasing $J_k \implies$ closed loop response does not follow predicted trajectory

Stability analysis

How can we guarantee the closed loop stability of MPC?

(a). Show that a Lyapunov function exists demonstrating stability
(b). Ensure that optimization feasible is at each time $k = 0, 1, \ldots$

- For Lyapunov stability analysis:
  - consider first the unconstrained problem
  - use predicted cost as a trial Lyapunov function

- Guarantee feasibility of the MPC optimization recursively
  by ensuring that feasibility at time k implies feasibility at $k+1$

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Discrete time Lyapunov stability

Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- Definition: $x = 0$ is a stable equilibrium point if

  $\max_k \|x_k\|$ can be made arbitrarily small

  by making x_0 sufficiently small

- If continuously differentiable $V(x)$ exists with

  (i). $V(x)$ is positive definite and

  (ii). $V(x_{k+1}) - V(x_k) \leq 0$

  then $x = 0$ is a stable equilibrium point

Discrete time Lyapunov stability

Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- Definition: $x = 0$ is a stable equilibrium point if

$$\text{for all } R > 0 \text{ there exists } r \text{ such that}$$
$$\|x_0\| < r \implies \|x_k\| < R \text{ for all } k$$

- If continuously differentiable $V(x)$ exists with

  (i). $V(x)$ is positive definite and

  (ii). $V(x_{k+1}) - V(x_k) \leq 0$

  then $x = 0$ is a stable equilibrium point

Discrete time Lyapunov stability

Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- Definition: $x = 0$ is an asymptotically stable equilibrium point if

  (i). $x = 0$ is stable and

  (ii). r exists such that $\|x_0\| < r \implies \lim_{k\to\infty} x_k = 0$

- If continuously differentiable $V(x)$ exists with

  (i). $V(x)$ is positive definite and

  (ii). $V(x_{k+1}) - V(x_k) < 0$ whenever $x_k \neq 0$

  then $x = 0$ is an asymptotically stable equilibrium point

Lyapunov stability

Trial Lyapunov function:

$$\boxed{J^*(x_k) = J(x_k, \mathbf{u}_k^*)}$$

where $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right)$

- ⋆ $J^*(x)$ is positive definite if:

  (a). $R \succeq 0$ and $Q \succ 0$, or
  (b). $R \succ 0$ and $Q \succeq 0$ and $(A, Q^{1/2})$ is observable

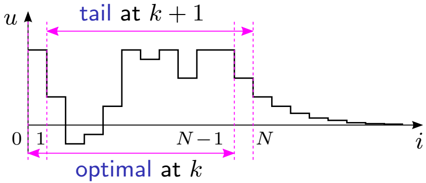
  since then $J^*(x_k) \geq 0$ and $J^*(x_k) = 0$ if and only if $x_k = 0$

- ⋆ $J^*(x)$ is continuously differentiable

  . . . from analysis of MPC optimization as a multiparametric QP

Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the "tail" of the optimal prediction at time k



Optimal predicted sequences at time k:

$$\mathbf{u}_k^* = \begin{bmatrix} u_{0|k}^* \\ u_{1|k}^* \\ \vdots \\ u_{N-1|k}^* \\ Kx_{N|k}^* \\ \vdots \end{bmatrix} \qquad \mathbf{x}_k^* = \begin{bmatrix} x_{0|k}^* \\ x_{1|k}^* \\ \vdots \\ x_{N|k}^* \\ \Phi x_{N|k}^* \\ \vdots \end{bmatrix}$$

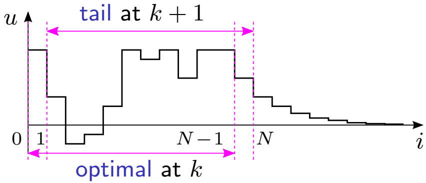
$(\Phi = A + BK)$

optimal at k : $\quad J^*(x_k) = J(x_k, \mathbf{u}_k^*) \quad = \sum_{i=0}^{\infty} \left(\|x_{i|k}^*\|_Q^2 + \|u_{i|k}^*\|_R^2 \right)$

tail at $k+1$: $\quad \tilde{J}(x_{k+1}) = J(x_{k+1}, \tilde{\mathbf{u}}_{k+1}) \quad = \sum_{i=1}^{\infty} \left(\|x_{i|k}^*\|_Q^2 + \|u_{i|k}^*\|_R^2 \right)$

Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the "tail" of the optimal prediction at time k



Tail sequences at time $k+1$:

$$\tilde{\mathbf{u}}_{k+1} = \begin{bmatrix} u^*_{1|k} \\ \vdots \\ u^*_{N-1|k} \\ Kx^*_{N|k} \\ K\Phi x^*_{N|k} \\ \vdots \end{bmatrix} \qquad \tilde{\mathbf{x}}_{k+1} = \begin{bmatrix} x^*_{1|k} \\ \vdots \\ x^*_{N|k} \\ \Phi x^*_{N|k} \\ \Phi^2 x^*_{N|k} \\ \vdots \end{bmatrix}$$

$(\Phi = A + BK)$

optimal at k: $\quad J^*(x_k) = J(x_k, \mathbf{u}_k^*) \quad = \sum_{i=0}^{\infty} \left(\|x^*_{i|k}\|_Q^2 + \|u^*_{i|k}\|_R^2 \right)$

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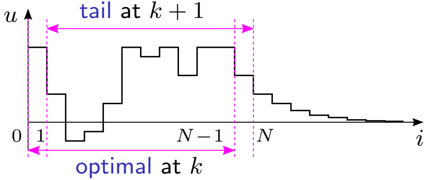
Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the "tail" of the optimal prediction at time k



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optimal at k : $\quad J^*(x_k) = J(x_k, \mathbf{u}^*_k) \quad = \sum_{i=0}^{\infty} \big(\|x^*_{i|k}\|^2_Q + \|u^*_{i|k}\|^2_R \big)$

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Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the "tail" of the optimal prediction at time k

Predicted cost for the tail:

$$\tilde{J}(x_{k+1}) = J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2$$

but $\tilde{\mathbf{u}}_{k+1}$ is suboptimal at time $k+1$, so

$$J^*(x_{k+1}) \leq \tilde{J}(x_{k+1})$$

Therefore

$$\boxed{J^*(x_{k+1}) \leq J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2}$$

Lyapunov stability

The bound $J^*(x_{k+1}) - J^*(x_k) \le -\|x_k\|_Q^2 - \|u_k\|_R^2$ implies:

(i). the closed loop cost cannot exceed the initial predicted cost, since summing both sides over all $k \ge 0$ gives

$$\sum_{k=0}^{\infty} \left(\|x_k\|_Q^2 + \|u_k\|_R^2\right) \le J^*(x_0)$$

(ii). $x = 0$ is asymptotically stable

- ⋆ if $R \succeq 0$ and $Q \succ 0$, this follows from Lyapunov's direct method
- ⋆ if $R \succ 0$, $Q \succeq 0$ and $(A, Q^{1/2})$ observable, this follows from:

  (a). stability of $x = 0$ $\Leftarrow$ Lyapunov's direct method

  (b). $\lim_{k\to\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2) = 0$ $\Leftarrow$ $\sum_{k=0}^{\infty}\left(\|x_k\|_Q^2 + \|u_k\|_R^2\right) < \infty$

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- For Lyapunov stability analysis:
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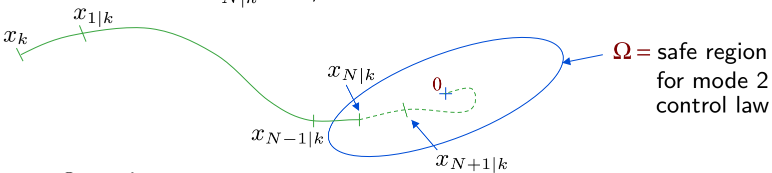
Terminal constraint

The basic idea

Terminal constraint

Terminal constraint: $x_{N|k} \in \Omega$, where $\Omega =$ terminal set



Choose Ω so that:

(a). $x \in \Omega \quad \Longrightarrow \quad \begin{cases} \underline{u} \leq Kx \leq \overline{u} \\ \underline{x} \leq x \leq \overline{x} \end{cases}$

(b). $x \in \Omega \quad \Longrightarrow \quad (A + BK)x \in \Omega$

then Ω is invariant for the mode 2 dynamics and constraints, so

$$x_{N|k} \in \Omega \implies \begin{cases} \underline{u} \leq u_{i|k} \leq \overline{u} \\ \underline{x} \leq x_{i|k} \leq \overline{x} \end{cases} \text{for } i = N, N+1, \ldots$$

i.e. constraints are satisfied over
the infinite mode 2 prediction horizon

Stability of constrained MPC

Prototype MPC algorithm

At each time $k = 0, 1, \ldots$

(i). solve $$\mathbf{u}_k^* = \arg\min_{\mathbf{u}_k} \; J(x_k, \mathbf{u}_k)$$
$$\begin{aligned} \text{s.t.} \;\; & \underline{u} \le u_{i|k} \le \overline{u}, \;\; i = 0, \ldots, N-1 \\ & \underline{x} \le x_{i|k} \le \overline{x}, \;\; i = 1, \ldots, N \\ & x_{N|k} \in \Omega \end{aligned}$$

(ii). apply $u_k = u^*_{0|k}$ to the system

Asymptotically stabilizes $x = 0$ with region of attraction $\mathcal{F}_N$,

$$\mathcal{F}_N = \left\{ x_0 : \exists \{u_0, \ldots, u_{N-1}\} \text{ such that } \begin{array}{l} \underline{u} \le u_i \le \overline{u}, \;\; i = 0, \ldots, N-1 \\ \underline{x} \le x_i \le \overline{x}, \;\; i = 1, \ldots, N \\ x_N \in \Omega \end{array} \right\}$$

$=$ the set of all feasible initial conditions for N-step horizon and terminal set Ω

Terminal constraints

Make Ω as large as possible so that the feasible set $\mathcal{F}_N$ is maximized, i.e.

$$\Omega = \mathcal{X}_\infty = \lim_{j\to\infty} \mathcal{X}_j$$

where

- $\mathcal{X}_j$ = initial conditions for which constraints are satisfied for j steps with $u = Kx$

$$= \left\{ x : \begin{array}{c} \underline{u} \leq K(A+BK)^i x \leq \overline{u} \\ \underline{x} \leq (A+BK)^i x \leq \overline{x} \end{array} \quad i = 0, \ldots, j \right\}$$

- $\mathcal{X}_\infty = \mathcal{X}_\nu$ for some finite ν if $|\mathrm{eig}(A+BK)| < 1$

$$\Downarrow$$

$x \in \mathcal{X}_\infty$ if constraints are satisfied on a finite constraint checking horizon

Terminal constraints – Example

Plant model:

$$x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k$$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

input constraints: $-1 \leq u_k \leq 1$

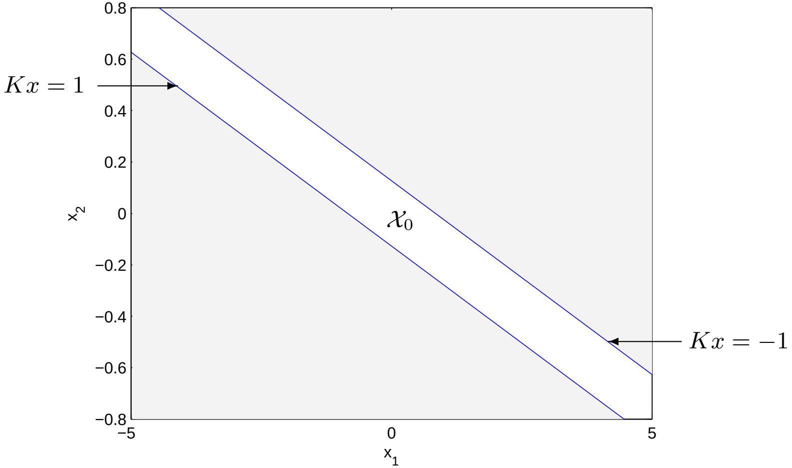
mode 2 feedback law:

$$K = \begin{bmatrix} -1.19 & -7.88 \end{bmatrix}$$

$$= K_{\mathsf{LQ}} \text{ for } Q = C^\top C, \; R = 1$$

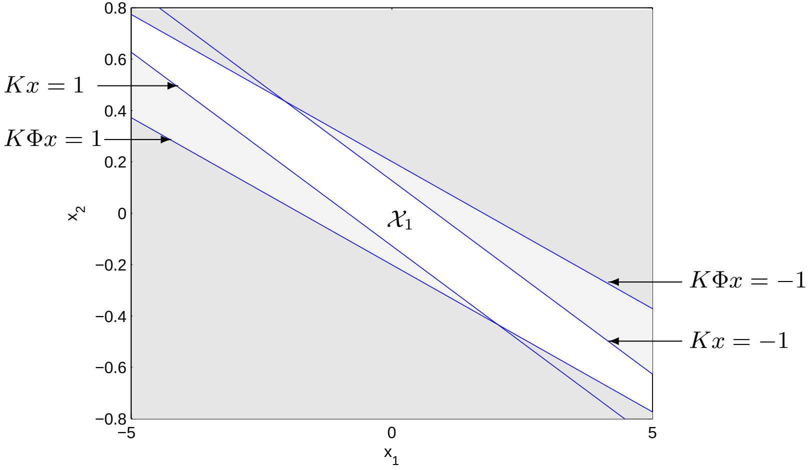
Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



Terminal constraints – example

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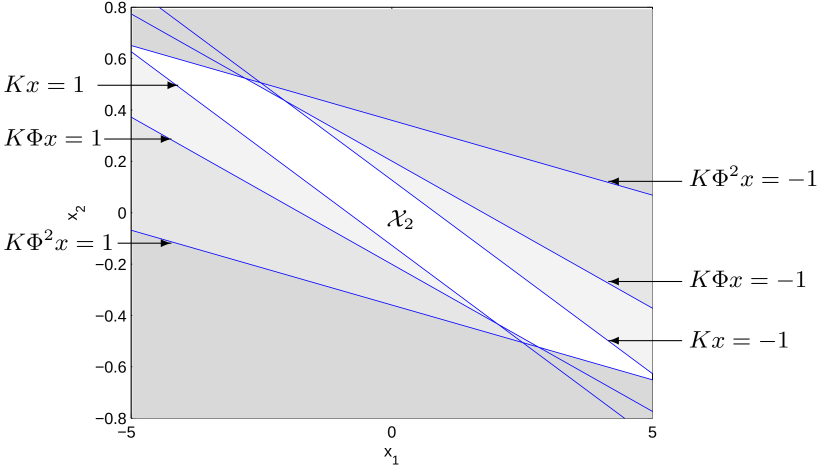
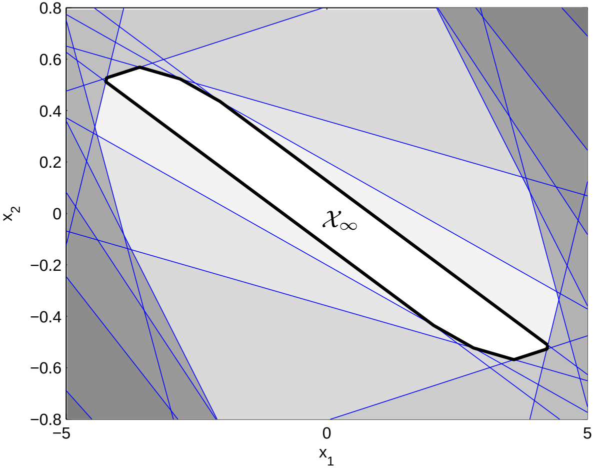


Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



$\mathcal{X}_4 = \mathcal{X}_5 = \cdots = \mathcal{X}_j$ for all $j > 4$ so $\mathcal{X}_4 = \mathcal{X}_\infty$

Terminal constraints – example

In this example $\mathcal{X}_\infty$ is determined in a finite number of steps because

- (A) $(A + BK)$ is strictly stable, and
- (B) $\big((A + BK), K\big)$ is observable

(A) $\Rightarrow \left\{ \begin{matrix} \text{shortest distance of hyperplane} \\ K(A+BK)^i x \le 1 \text{ from origin} \end{matrix} \right\} = \dfrac{1}{\|K(A+BK)^i\|_2}$

$\rightarrow \infty \quad \text{as } i \rightarrow \infty$

(B) $\Rightarrow \mathcal{X}_\infty$ is bounded because $x_0 \notin \mathcal{X}_\infty$ if x_0 is sufficiently large

Here $\{x : -1 \le K(A+BK)^i x \le 1\}$ contains $\mathcal{X}_4$ for $i > 4$

$$\Downarrow$$

$$\mathcal{X}_\infty = \mathcal{X}_4$$

constraint checking horizon: $\nu = 4$

Terminal constraints – example

In this example $\mathcal{X}_\infty$ is determined in a finite number of steps because

- (A) $(A + BK)$ is strictly stable, and
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$$\to \infty \quad \text{as } i \to \infty$$

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Here $\{x : -1 \le K(A+BK)^i x \le 1\}$ contains $\mathcal{X}_4$ for $i > 4$

$$\Downarrow$$

$$\mathcal{X}_\infty = \mathcal{X}_4$$

constraint checking horizon: $\nu = 4$

Terminal constraints

General case

Let $\mathcal{X}_j = \{x : F\Phi^i x \le \mathbf{1},\ i = 0, \ldots j\}$ with $\begin{cases} \Phi \text{ strictly stable} \\ (\Phi, F) \text{ observable} \end{cases}$

then:
(i). $\mathcal{X}_\infty = \mathcal{X}_\nu$ for finite ν
(ii). $\mathcal{X}_\nu = \mathcal{X}_\infty$ iff $x \in \mathcal{X}_{\nu+1}$ whenever $x \in \mathcal{X}_\nu$

Proof of (ii)

(a). for any j, $\mathcal{X}_{j+1} = \mathcal{X}_j \cap \big\{x : F\Phi^{j+1}x \le \mathbf{1}\big\}$

so $\mathcal{X}_j \supseteq \mathcal{X}_{j+1} \supseteq \lim_{j\to\infty} \mathcal{X}_j = \mathcal{X}_\infty$

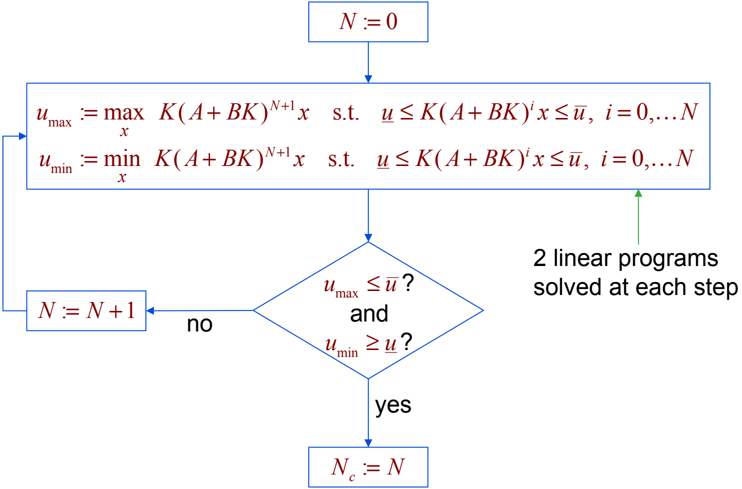
(b). if $x \in \mathcal{X}_{\nu+1}$ whenever $x \in \mathcal{X}_\nu$, then $\Phi x \in \mathcal{X}_\nu$ whenever $x \in \mathcal{X}_\nu$

but $\mathcal{X}_\nu \subseteq \big\{x : Fx \le \mathbf{1}\big\}$ and it follows that $\mathcal{X}_\nu \subseteq \mathcal{X}_\infty$

(a) & (b) $\Rightarrow$ $\mathcal{X}_\nu = \mathcal{X}_\infty$

Terminal constraints – constraint checking horizon

Algorithm for computing constraint checking horizon N_c
for input constraints $\underline{u} \leq u \leq \overline{u}$:



Constrained MPC

Define the terminal set Ω as $\mathcal{X}_{N_c}$

MPC algorithm

At each time $k = 0, 1, \ldots$

(i). solve
$$\begin{aligned} \mathbf{u}_k^* = \arg\min_{\mathbf{u}_k} \;\; & J(x_k, \mathbf{u}_k) \\ \text{s.t.} \;\; & \underline{u} \leq u_{i|k} \leq \overline{u}, \;\; i = 0, \ldots, N + N_c \\ & \underline{x} \leq x_{i|k} \leq \overline{x}, \;\; i = 1, \ldots, N + N_c \end{aligned}$$

(ii). apply $u_k = u_{0|k}^*$ to the system

Note

- ⋆ predictions for $i = N, \ldots N + N_c$: $\begin{cases} x_{i|k} = (A + BK)^{i-N} x_{N|k} \\ u_{i|k} = K(A + BK)^{i-N} x_{N|k} \end{cases}$
- ⋆ $x_{N|k} \in \mathcal{X}_{N_c}$ implies linear constraints so online optimization is a QP

Closed loop performance

Longer horizon N ensures improved predicted cost $J^*(x_0)$

and is likely (but not certain) to give better closed-loop performance

Example: Cost vs N for $x_0 = (-7.5, 0.5)$

N	6	7	8	11	> 11
$J^*(x_0)$	364.2	357.0	356.3	356.0	356.0
$J_{\text{cl}}(x_0)$	356.0	356.0	356.0	356.0	356.0

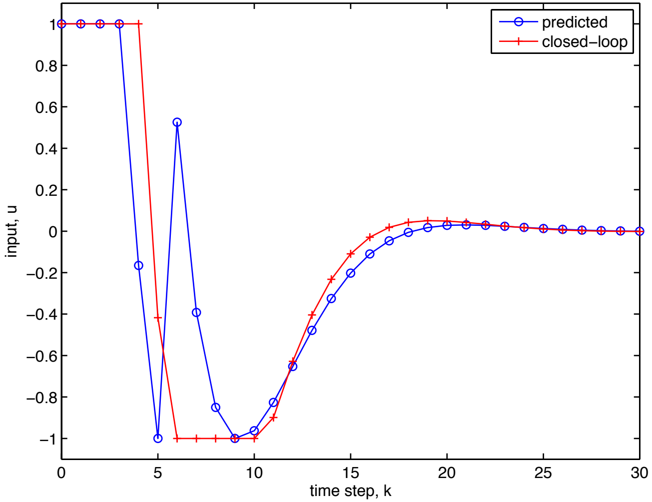
Closed loop cost: $J_{\text{cl}}(x_0) := \sum_{k=0}^{\infty} \big(\|x_k\|_Q^2 + \|u_k\|_R^2 \big)$

For this initial condition:

MPC with $N = 11$ is identical to constrained LQ optimal control ($N = \infty$)!

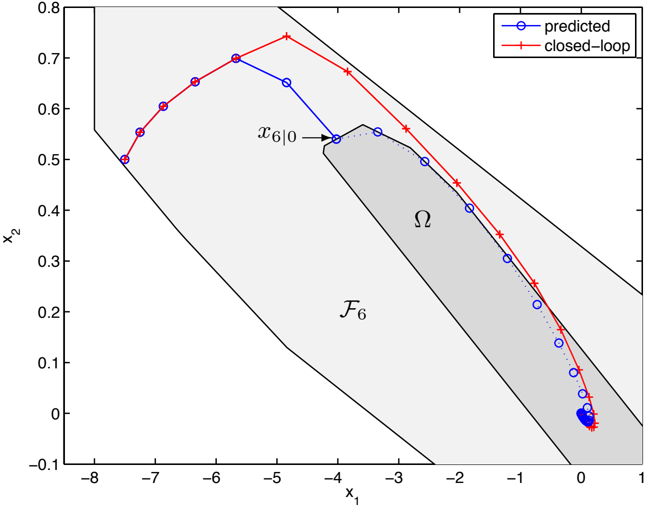
Closed loop performance – example

Predicted and closed loop inputs for $N = 6$



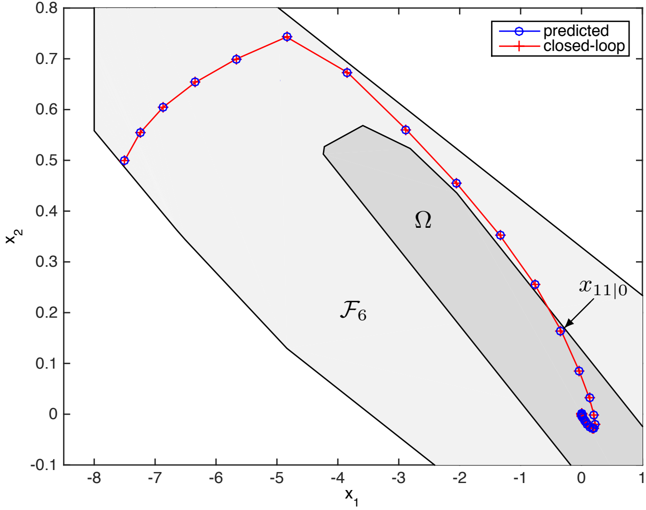
Closed loop performance – example

Predicted and closed loop states for $N = 6$



Closed loop performance – example

Predicted and closed loop states for $N = 11$



Choice of mode 1 horizon – performance

- For this x_0: $N = 11 \Rightarrow x_{N|0}$ lies in the interior of Ω

$$\Updownarrow$$

terminal constraint is inactive

$$\Downarrow$$

no reduction in cost for $N > 11$

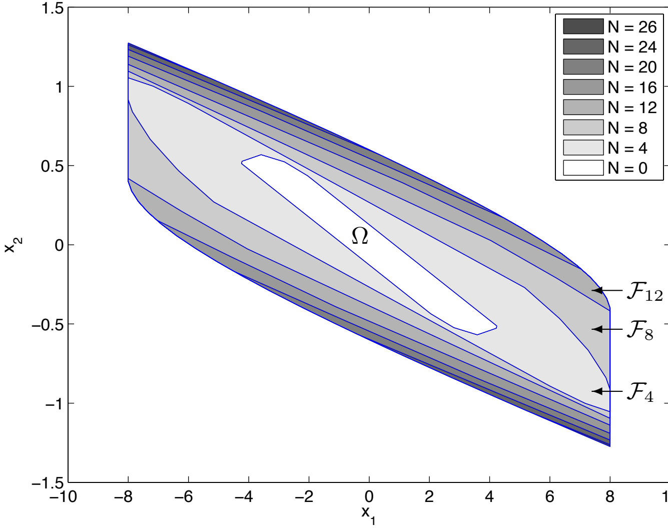
- Constrained LQ optimal performance is always obtained with $N \geq N_\infty$ for some finite N_∞ dependent on x_0

- N_∞ may be large, implying high computational load but closed loop performance is often close to optimal for $N < N_\infty$ (due to receding horizon)

  in this example $J_{\text{cl}}(x_0) \approx$ optimal for $N \geq 6$

Choice of mode 1 horizon – region of attraction

Increasing N increases the feasible set $\mathcal{F}_N$



Summary

- Linear MPC ingredients:
  - ⋆ Infinite cost horizon (via terminal cost)
  - ⋆ Terminal constraints (via constraint-checking horizon)
- Constraints are satisfied over an infinite prediction horizon
- Closed-loop system is asymptotically stable
  with region of attraction equal to the set of feasible initial conditions
- Ideal optimal performance if mode 1 horizon N is large enough

Lecture 4

Robustness to disturbances

Robustness to disturbances

- Review of nominal model predictive control
- Setpoint tracking and integral action
- Robustness to unknown disturbances
- Handling time-varying disturbances

Review

MPC with guaranteed stability – the basic idea

Review

MPC optimization for linear model $x_{k+1} = Ax_k + Bu_k$

$$\begin{aligned} \underset{\mathbf{u}_k}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) + \|x_{N|k}\|_P^2 \\ \text{subject to} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \ \ i = 0, \ldots, N + N_c \\ & \underline{x} \le x_{i|k} \le \overline{x}, \ \ i = 1, \ldots, N + N_c \end{aligned}$$

where

- $u_{i|k} = Kx_{i|k}$ for $i \ge N$, with K = unconstrained LQ optimal
- terminal cost: $\|x_{N|k}\|_P^2 = \sum_{i=N}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2)$, with

$$P - \Phi^T P \Phi = Q + K^T R K, \quad \Phi = A + BK$$

- terminal constraints are defined by the constraint checking horizon N_c:

$$\left.\begin{array}{c} \underline{u} \le K\Phi^i x \le \overline{u} \\ \underline{x} \le \Phi^i x \le \overline{x} \end{array}\right\} \ i = 0, \ldots, N_c \implies \left\{\begin{array}{c} \underline{u} \le K\Phi^{N_c+1} x \le \overline{u} \\ \underline{x} \le \Phi^{N_c+1} x \le \overline{x} \end{array}\right.$$

Review

MPC optimization for linear model $x_{k+1} = Ax_k + Bu_k$

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Review

MPC optimization for nonlinear model $x_{k+1} = f(x_k, u_k)$

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with

- ⋆ mode 2 feedback: $u_{i|k} = \kappa(x_{i|k})$ asymptotically stabilizes $x = 0$ (locally)
- ⋆ terminal cost: $\|x_{N|k}\|_P^2 \ge \sum_{i=N}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right)$

  for mode 2 dynamics: $x_{i+1|k} = f\big(x_{i|k}, \kappa(x_{i|k})\big)$
- ⋆ terminal constraint set Ω: invariant for mode 2 dynamics and constraints

$$\left.\begin{array}{c} f\big(x, \kappa(x)\big) \in \Omega \\ \underline{u} \le \kappa(x) \le \overline{u}, \ \ \underline{x} \le x \le \overline{x} \end{array}\right\} \quad \text{for all } x \in \Omega$$

Review

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& x_{N|k} \in \Omega
\end{aligned}
$$

with

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$$

Comparison

- Linear MPC

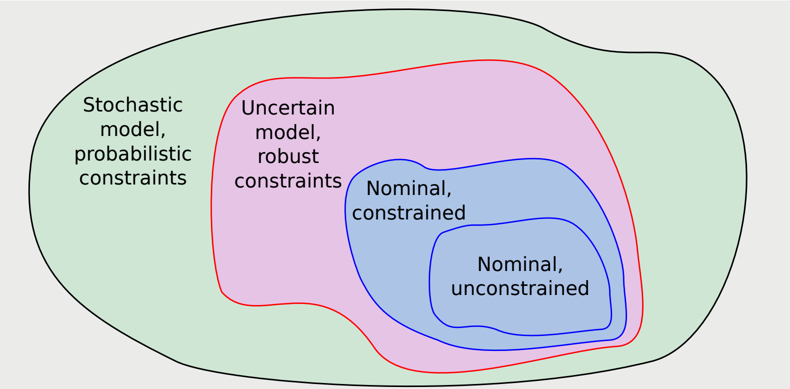
terminal cost	$\longleftarrow$	exact cost over the mode 2 horizon
terminal constraint set	$\longleftarrow$	contains all feasible initial conditions for mode 2

- Nonlinear MPC

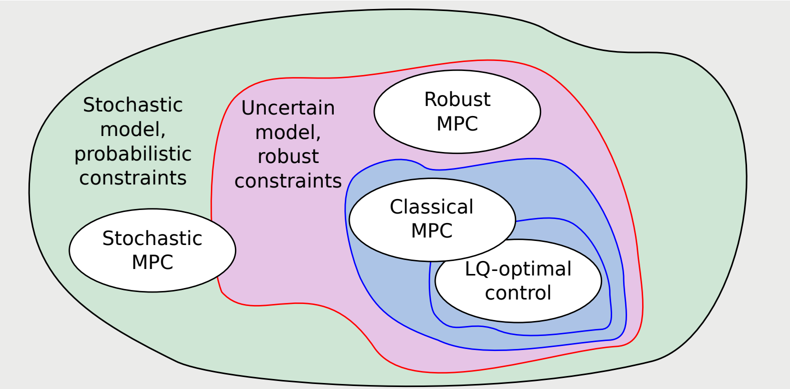
terminal cost	$\longleftarrow$	upper bound on cost over mode 2 horizon
terminal constraint set	$\longleftarrow$	invariant set (usually not the largest) for mode 2 dynamics and constraints

Model uncertainty

Model uncertainty



Model uncertainty



Model uncertainty

Common causes of model error and uncertainty:

- Unknown or time-varying model parameters
  - unknown loads & inertias, static friction
  - unknown d.c. gain
- Random (stochastic) model parameters
  - random process noise or sensor noise
- Incomplete measurement of states
  - state estimation error

Setpoint tracking

- Output setpoint: y^0

$$y \to y^0 \quad \Rightarrow \quad \begin{cases} x \to x^0 \\ u \to u^0 \end{cases} \quad \text{where} \quad \begin{aligned} x^0 &= Ax^0 + Bu^0 \\ y^0 &= Cx^0 \end{aligned}$$

$$\Downarrow$$

$$y^0 = C(I - A)^{-1}Bu^0$$

- Setpoint for (u^0, x^0) is unique iff $C(I - A)^{-1}B$ is invertible

e.g. if $\dim(u) = \dim(y)$, then $\begin{cases} u^0 = \big(C(I - A)^{-1}B\big)^{-1}y^0 \\ x^0 = (I - A)^{-1}Bu^0 \end{cases}$

- Tracking problem: $y_k \to y^0$ subject to $\begin{cases} \underline{u} \leq u_k \leq \overline{u} \\ \underline{x} \leq x_k \leq \overline{x} \end{cases}$

is only feasible if $\underline{u} \leq u^0 \leq \overline{u}$ and $\underline{x} \leq x^0 \leq \overline{x}$

Setpoint tracking

- Unconstrained tracking problem:

$$\underset{\mathbf{u}_k^\delta}{\text{minimize}} \quad \sum_{i=0}^{\infty} \left(\|x_{i|k}^\delta\|_Q^2 + \|u_{i|k}^\delta\|_R^2 \right)$$

where $x^\delta = x - x^0$
$u^\delta = u - u^0$

has optimal solution: $u_k = K x_k^\delta + u^0, \quad K = K_{\mathsf{LQ}}$

- Constrained tracking problem:

$$\begin{aligned} \underset{\mathbf{u}_k^\delta}{\text{minimize}} \quad & \sum_{i=0}^{\infty} \left(\|x_{i|k}^\delta\|_Q^2 + \|u_{i|k}^\delta\|_R^2 \right) \\ \text{subject to} \quad & \underline{u} \le u_{i|k}^\delta + u^0 \le \overline{u}, \qquad i = 0, 1, \ldots \\ & \underline{x} \le x_{i|k}^\delta + x^0 \le \overline{x}, \qquad i = 1, 2, \ldots \end{aligned}$$

has optimal solution: $u_k = u_{0|k}^{\delta *} + u^0$

Setpoint tracking

If $\hat{u}^0$ is used instead of u^0 (e.g. if d.c. gain $C(I-A)^{-1}B$ unknown)

then $u_k = u^{\delta *}_{0|k} + \hat{u}^0$ implies

$$\begin{aligned} u^\delta_k &= u^{\delta *}_{0|k} + (\hat{u}^0 - u^0) \\ x^\delta_{k+1} &= Ax^\delta_k + Bu^{\delta *}_{0|k} + B\underbrace{(\hat{u}^0 - u^0)}_{\text{constant disturbance}} \end{aligned}$$

and if $u^{\delta *}_{0|k} \to Kx^\delta_k$ as $k \to \infty$, then

$$\begin{aligned} \lim_{k\to\infty} x^\delta_k &= (I - A - BK)^{-1}B(\hat{u}^0 - u^0) \qquad \neq 0 \\ \lim_{k\to\infty} y_k - y^0 &= \underbrace{C(I - A - BK)^{-1}B(\hat{u}^0 - u^0)}_{\text{steady state tracking error}} \quad \neq 0 \end{aligned}$$

Additive disturbances

Convert (constant) setpoint tracking problem into a regulation problem:

$$x \leftarrow x^\delta,\ y \leftarrow y^\delta,\ u \leftarrow u^\delta$$

Consider the effect of additive disturbance w:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Dw_k, \\ y_k &= Cx_k \end{aligned}$$

Assume that w_k is unknown at time k, but is known to be:

- ⋆ constant ($w_k = w$ for all k) or time-varying
- ⋆ within a known polytopic set: $w_k \in \mathcal{W}$ for all k

where $\mathcal{W} = \operatorname{conv}\{w^{(1)}, \ldots, w^{(r)}\}$

or $\mathcal{W} = \{w : Hw \leq \mathbf{1}\}$

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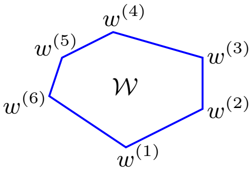

Integral action (no constraints)

Introduce integral action to remove steady state error in y
by considering the augmented system:

$$z_k = \begin{bmatrix} x_k \\ v_k \end{bmatrix}, \qquad z_{k+1} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} z_k + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} D \\ 0 \end{bmatrix} w_k$$

$v_k =$ integrator state

$v_{k+1} = v_k + y_k$

⋆ Linear feedback $\quad u_k = Kx_k + K_I v_k$

is stabilizing if $\quad \left| \operatorname{eig}\left(\begin{bmatrix} A + BK & BK_I \\ C & I \end{bmatrix} \right) \right| < 1$

⋆ If the closed-loop system is (strictly) stable and $w_k \to w =$ constant

then $\quad u_k \to u^{ss} \quad \Longrightarrow \quad v_k \to v^{ss} \quad \Longrightarrow \quad y_k \to 0 \quad$ even if $w \neq 0$

but arbitrary K_I may destabilize the closed loop system

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but arbitrary K_I may destabilize the closed loop system

Integral action (no constraints)

Ensure stability by using a modified cost:

$$\underset{\mathbf{u}_k}{\text{minimize}} \quad \sum_{i=0}^{\infty} \left(\|z_{i|k}\|_{Q_z}^2 + \|u_{i|k}\|_R^2 \right) \qquad Q_z = \begin{bmatrix} Q & 0 \\ 0 & Q_I \end{bmatrix} \succeq 0$$

with predictions generated by an augmented model

$$z_{i+1|k} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} z_{i|k} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{i|k}, \qquad z_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}$$

- ⋆ this is a "nominal" prediction model since $w_k = 0$ is assumed
- ⋆ unconstrained solution: $u_k = K_z z_k = K x_k + K_I v_k$
- ⋆ if $R \succ 0$, $\left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix}, \begin{bmatrix} Q & 0 \\ 0 & Q_I \end{bmatrix} \right)$ is observable and $w_k \to w =$ constant

  then $\quad u_k \to u^{ss} \quad \Longrightarrow \quad v_k \to v^{ss} \quad \Longrightarrow \quad y_k \to 0$

Integral action – example

Plant model:

$$x_{k+1} = Ax_k + Bu_k + Dw \qquad\qquad y_k = Cx_k$$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

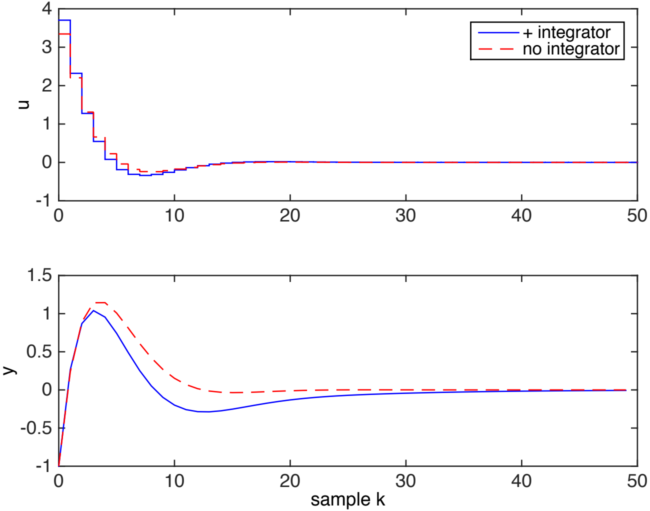
Constraints: none

Cost weighting matrices: $Q_z = \begin{bmatrix} C^T C & 0 \\ 0 & 0.01 \end{bmatrix}$, $R = 1$

Unconstrained LQ optimal feedback gain:

$$K_z = \begin{bmatrix} -1.625 & -9.033 & 0.069 \end{bmatrix}$$

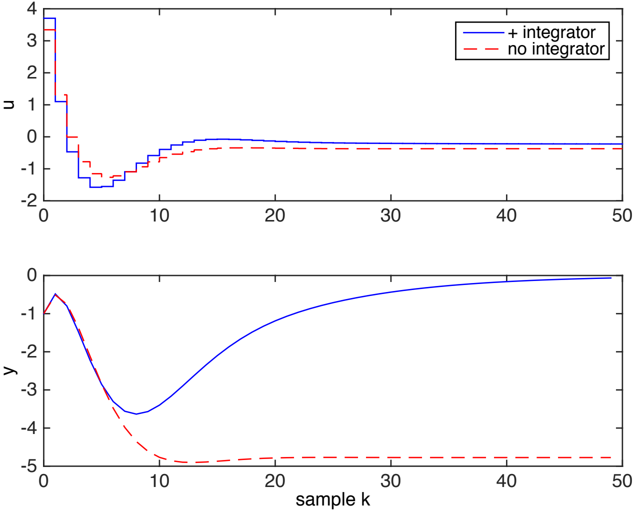
Integral action – example



Closed loop response for initial condition: $x_0 = (0.5, -0.5)$

no disturbance: $w = 0$

Integral action – example



Closed loop response for initial condition: $x_0 = (0.5, -0.5)$

constant disturbance: $w = 0.75$

Constrained MPC

Naive constrained MPC strategy: $w = 0$ assumed in predictions

$$\begin{aligned} \underset{\mathbf{u}_k}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|z_{i|k}\|^2_{Q_z} + \|u_{i|k}\|^2_R \right) + \|z_{N|k}\|^2_P \\ \text{subject to} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \;\; i = 0, \ldots, N + N_c \\ & \underline{x} \le x_{i|k} \le \overline{x}, \;\; i = 1, \ldots, N + N_c \end{aligned}$$

with: P and N_c determined for mode 2 control law $u_{i|k} = K_z z_{i|k}$

and initial prediction state: $z_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}$ where $v_{k+1} = v_k + y_k$

⋆ If closed loop system is stable

then $\quad u_k \to u^{ss} \quad \Longrightarrow \quad v_k \to v^{ss} \quad \Longrightarrow \quad y_k \to 0$

⋆ but disturbance w_k is ignored in predictions, so

$$\begin{cases} J^*(z_{k+1}) - J^*(z_k) \not\le 0 \\ \text{feasibility at time } k \not\Rightarrow \text{ feasibility at } k+1 \end{cases}$$

therefore no guarantee of stability

Constrained MPC

Naive constrained MPC strategy: $w = 0$ assumed in predictions

$$\begin{aligned} \underset{\mathbf{u}_k}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|z_{i|k}\|^2_{Q_z} + \|u_{i|k}\|^2_R \right) + \|z_{N|k}\|^2_P \\ \text{subject to} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \ \ i = 0, \ldots, N + N_c \\ & \underline{x} \le x_{i|k} \le \overline{x}, \ \ i = 1, \ldots, N + N_c \end{aligned}$$

with: P and N_c determined for mode 2 control law $u_{i|k} = K_z z_{i|k}$

and initial prediction state: $z_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}$ where $v_{k+1} = v_k + y_k$

- ⋆ If closed loop system is stable

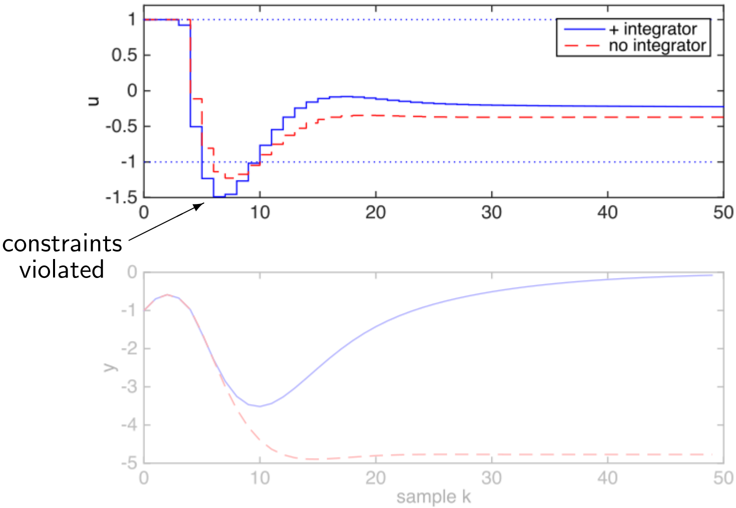
  then $\quad u_k \to u^{ss} \quad \Longrightarrow \quad v_k \to v^{ss} \quad \Longrightarrow \quad y_k \to 0$

- ⋆ but disturbance w_k is ignored in predictions, so

$$\begin{cases} J^*(z_{k+1}) - J^*(z_k) \not\le 0 \\ \text{feasibility at time } k \not\Rightarrow \text{ feasibility at } k+1 \end{cases}$$

therefore no guarantee of stability

Constrained MPC – example



Closed loop response with
constraints: $-1 \leq u \leq 1$

initial condition: $x_0 = (0.5, -0.5)$
disturbance: $w = 0.75$

Robust constraints

If predictions satisfy constraints $\begin{cases} \text{for all prediction times } i = 0, 1, \ldots \\ \text{for all disturbances } w_i \in \mathcal{W} \end{cases}$

then feasibility of constraints at time k ensures feasibility at time $k+1$

- Decompose predictions into

  nominal predicted state $s_{i|k}$
  uncertain predicted state $e_{i|k}$

  where

$$x_{i|k} = s_{i|k} + e_{i|k} \qquad \begin{cases} s_{i+1|k} = \Phi s_{i|k} + B c_{i|k} & s_{0|k} = x_k \\ e_{i+1|k} = \Phi e_{i|k} + D w_{i|k} & e_{0|k} = 0 \end{cases}$$

- Pre-stabilized predictions:

$$u_{i|k} = K x_{i|k} + c_{i|k} \text{ and } \Phi = A + BK$$

  where $K = K_{\mathsf{LQ}}$ is the unconstrained LQ optimal gain

Pre-stabilized predictions – example

Scalar system:	$x_{k+1} = 2x_k + u_k + w_k$,	constraint:	$\lvert x_k \rvert \le 2$
uncertainty:	$e_{i\|k} = \sum_{j=0}^{i-1} 2^j w = (2^i - 1)w$,	disturbance:	$w_k = w$ $\lvert w \rvert \le 1$

Pre-stabilized predictions – example

Scalar system: $x_{k+1} = 2x_k + u_k + w_k$, constraint: $|x_k| \le 2$

uncertainty: $e_{i|k} = \sum_{j=0}^{i-1} 2^j w = (2^i - 1)w$, disturbance: $w_k = w$, $|w| \le 1$

Robust constraints:

$$|s_{i|k} + e_{i|k}| \le 2 \quad \text{for all } |w| \le 1$$

$$\Updownarrow$$

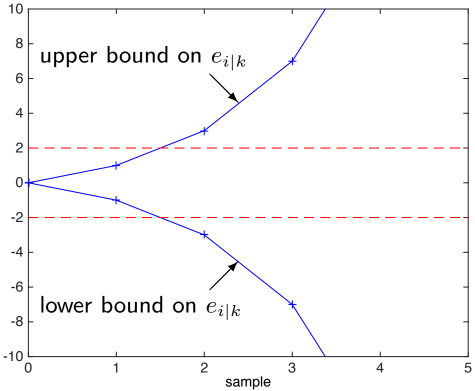
$$|s_{i|k}| \le 2 - \max_{|w|\le 1} |e_{i|k}|$$

$$\Downarrow$$

$$|s_{i|k}| \le 2 - (2^i - 1)$$

$$\Downarrow$$

infeasible for all $i > 1$



Pre-stabilized predictions – example

Avoid infeasibility by using pre-stabilized predictions:

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \qquad K = -1.9, \qquad c_{i|k} = \begin{cases} \text{free} & i = 0, \ldots, N-1 \\ 0 & i \geq N \end{cases}$$

stable predictions: $e_{i|k} = \displaystyle\sum_{j=0}^{i-1} 0.1^j w = (1 - 0.1^i)w/0.9, \quad |w| \leq 1$

Pre-stabilized predictions – example

Avoid infeasibility by using pre-stabilized predictions:

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Robust constraints:

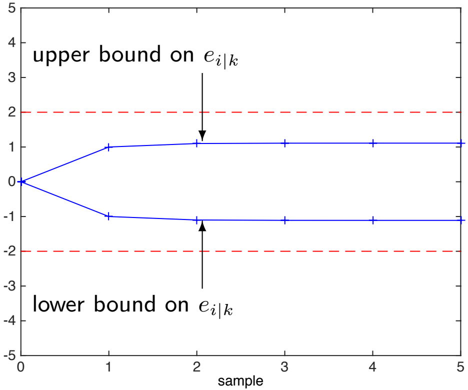
$|s_{i|k} + e_{i|k}| \leq 2$ for all $|w| \leq 1$

$\Updownarrow$

$|s_{i|k}| \leq 2 - \max_{|w|\leq 1} |e_{i|k}|$

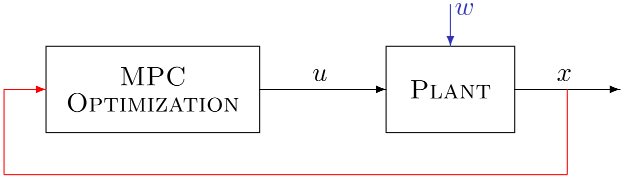
$\Downarrow$

$|s_{i|k}| \leq \underbrace{2 - (1 - 0.1^i)/0.9}_{>0 \text{ for all } i}$

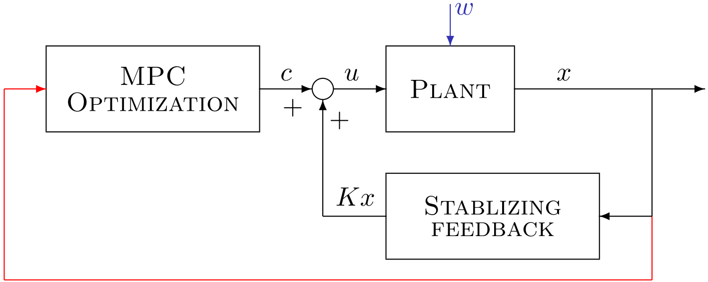


Pre-stabilized predictions

▷ Feedback structure of MPC with open loop predictions:



▷ Feedback structure of MPC with pre-stabilized predictions:



General form of robust constraints

How can we impose (general linear) constraints robustly?

⋆ Pre-stabilized predictions:

$$x_{i|k} = s_{i|k} + e_{i|k} \qquad \begin{cases} s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k} & s_{0|k} = x_k \\ e_{i+1|k} = \Phi e_{i|k} + Dw_{i|k} & e_{0|k} = 0 \end{cases}$$

$$\Longrightarrow \quad e_{i|k} = Dw_{i-1} + \Phi Dw_{i-2} + \cdots + \Phi^{i-1} Dw_0$$

⋆ General linear constraints: $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$

are equivalent to tightened constraints on nominal predictions:

$$\boxed{(F + GK)s_{i|k} + Gc_{i|k} \leq \mathbf{1} - h_i}$$

where $\quad h_0 = 0$

$$h_i = \max_{w_0,\ldots,w_{i-1} \in \mathcal{W}} (F + GK)e_{i|k}, \quad i = 1, 2, \ldots$$

(i.e. $h_i = h_{i-1} + \max_{w \in \mathcal{W}} (F + GK)w$
requiring one LP for each row of h_i)

Tube interpretation

The uncertainty in predictions: $e_{i+1|k} = \Phi e_{i|k} + D w_i, \; w_i \in \mathcal{W}$

evolves inside a tube (a sequence of sets): $e_{i|k} \in E_{i|k}$, where

$$E_{i|k} = D\mathcal{W} \oplus \Phi D\mathcal{W} \oplus \cdots \oplus \Phi^{i-1} D\mathcal{W}, \quad i = 1, 2, \ldots$$

Hence we can define:

- ⋆ a state tube $x_{i|k} = s_{i|k} + e_{i|k} \in \mathcal{X}_{i|k}$

$$\mathcal{X}_{i|k} = \{s_{i|k}\} \oplus E_{i|k}, \; i = 0, 1, \ldots$$

- ⋆ a control input tube $u_{i|k} = K x_{i|k} + c_{i|k} = K s_{i|k} + c_{i|k} + K e_{i|k} \in \mathcal{U}_{i|k}$

$$\mathcal{U}_{i|k} = \{K s_{i|k} + c_{i|k}\} \oplus K E_{i|k}, \; i = 0, 1, \ldots$$

and impose constraints robustly for the state and input tubes

(where $\oplus$ is Minkowski set addition)

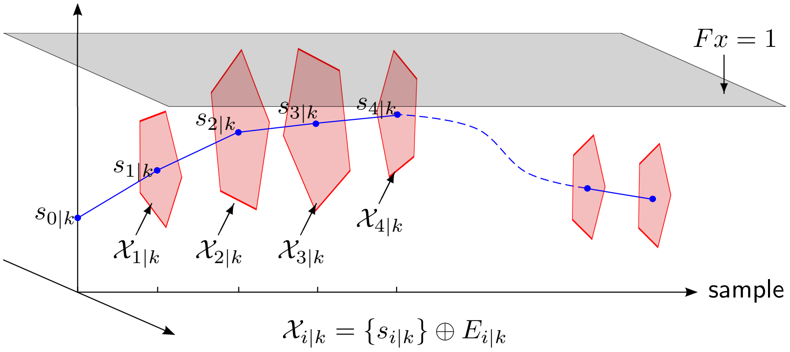
Tube interpretation

The uncertainty in predictions: $e_{i+1|k} = \Phi e_{i|k} + Dw_i$, $w_i \in \mathcal{W}$

evolves inside a tube (a sequence of sets): $e_{i|k} \in E_{i|k}$, where

$$E_{i|k} = D\mathcal{W} \oplus \Phi D\mathcal{W} \oplus \cdots \oplus \Phi^{i-1} D\mathcal{W}, \quad i = 1, 2, \ldots$$

e.g. for constraints $Fx \leq \mathbf{1}$ $(G = 0)$



Robust MPC

Prototype robust MPC algorithm

Offline: compute N_c and $h_1, \ldots, h_{N_c}$. Online at $k = 0, 1, \ldots$:

(i). solve $$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \ J(x_k, \mathbf{c}_k)$$
$$\text{s.t.} \ \ (F + GK)s_{i|k} + Gc_{i|k} \leq \mathbf{1} - h_i, \ i = 0, \ldots, N + N_c$$

(ii). apply $u_k = Kx_k + c_{0|k}^*$ to the system

- ⋆ tightened linear constraints are applied to nominal predictions
- ⋆ N_c is the constraint-checking horizon defined by:

  $$(F + GK)\Phi^{N_c+1}s \leq \mathbf{1} - h_{N_c+1}$$

  for all s satisfying $(F + GK)\Phi^i s \leq \mathbf{1} - h_i, \quad i = 0, \ldots, N_c$
- ⋆ the online optimization is robustly recursively feasible

Robust MPC

Prototype robust MPC algorithm

Offline: compute N_c and $h_1, \ldots, h_{N_c}$. Online at $k = 0, 1, \ldots$:

(i). solve
$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \; J(x_k, \mathbf{c}_k)$$
$$\text{s.t.} \;\; (F + GK)s_{i|k} + Gc_{i|k} \leq \mathbf{1} - h_i, \;\; i = 0, \ldots, N + N_c$$

(ii). apply $u_k = Kx_k + c^*_{0|k}$ to the system

nominal cost, evaluated assuming $w_i = 0$ for all i:

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \left(\|s_{i|k}\|_Q^2 + \|Ks_{i|k} + c_{i|k}\|_R^2 \right) = \|x_k\|_P^2 + \|\mathbf{c}_k\|_{W_c}^2$$

(one possible choice)

Convergence of robust MPC with nominal cost

If $u_{i|k} = Kx_{i|k} + c_{i|k}$ for $K = K_{\mathsf{LQ}}$, then:

- $\star$ the unconstrained solution is $\mathbf{c}_k = 0$, so the nominal cost is

$$J(x_k, \mathbf{c}_k) = \|x_k\|_P^2 + \|\mathbf{c}_k\|_{W_c}^2$$

and W_c is block-diagonal: $W_c = \operatorname{diag}\{P_c, \ldots, P_c\}$

- $\star$ recursive feasibility $\Rightarrow$ $\tilde{\mathbf{c}}_{k+1} = (c^*_{1|k}, \ldots, c^*_{N-1|k}, 0)$ feasible at $k+1$

- $\star$ hence $\quad \|\mathbf{c}^*_{k+1}\|^2_{W_c} \leq \|\mathbf{c}^*_k\|^2_{W_c} - \|c^*_{0|k}\|^2_{P_c}$

$$\Rightarrow \quad \sum_{k=0}^{\infty} \|c_{0|k}\|^2_{P_c} \leq \|\mathbf{c}^*_0\|^2_{W_c} < \infty$$

$$\Rightarrow \quad \lim_{k\to\infty} c_{0|k} = 0$$

- $\star$ therefore $\quad u_k \to Kx_k$ as $k \to \infty$

  $x_k \to$ the (minimal) robustly invariant set under unconstrained LQ optimal feedback

Robust MPC with constant disturbance

Assume $\quad w_k = w = \text{constant for all } k$

combine: pre-stabilized predictions
augmented state space model

⋆ Predicted state and input sequences:

$$x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} (s_{i|k} + e_{i|k})$$
$$u_{i|k} = K_z(s_{i|k} + e_{i|k}) + c_{i|k}$$

⋆ Prediction model:

nominal $\quad s_{i+1|k} = \Phi s_{i|k} + \begin{bmatrix} B \\ 0 \end{bmatrix} c_{i|k} \qquad \Phi = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_z$

uncertain $\quad e_{i|k} = \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w \qquad s_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}, \; e_{0|k} = 0$

⋆ Nominal cost:

$$J(x_k, v_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \big(\|s_{i|k}\|^2_{Q_z} + \|K_z s_{i|k} + c_{i|k}\|^2_R \big)$$

Robust MPC with constant disturbance

Assume $w_k = w =$ constant for all k

combine: pre-stabilized predictions
augmented state space model

- robust state constraints:

$$\underline{x} \le x_{i|k} \le \overline{x} \quad \Longleftrightarrow \quad \underline{x} + h_i \le s_{i|k} \le \overline{x} - h_i$$

$$h_i = \max_{w \in \mathcal{W}} \begin{bmatrix} I & 0 \end{bmatrix} \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w$$

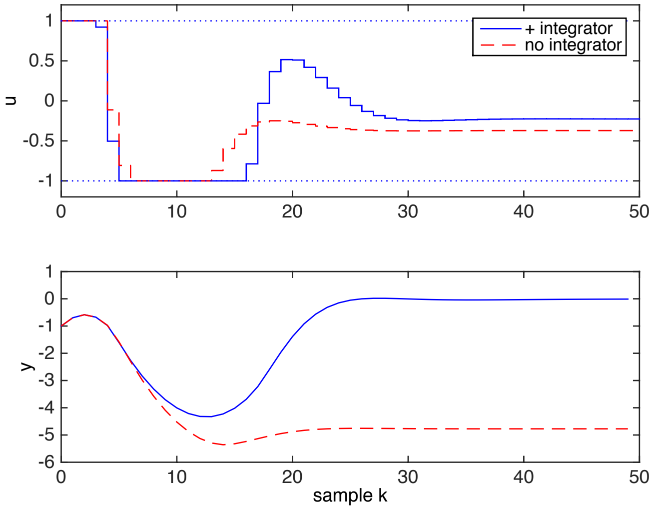
- robust input constraints:

$$\underline{u} \le u_{i|k} \le \overline{u} \quad \Longleftrightarrow \quad \underline{u} + h'_i \le K_z s_{i|k} + c_{i|k} \le \overline{u} - h'_i$$

$$h'_i = \max_{w \in \mathcal{W}} K_z \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w$$

- N_c and h_i, h'_i for $i = 1, \ldots, N_c$ can be computed offline

Robust MPC with constant disturbance – example



Closed loop response with initial condition: $x_0 = (0.5, -0.5)$

constraints: $-1 \leq u \leq 1$ disturbance: $w = 0.75$

Summary

- Integral action: augment model with integrated output error
  include integrated output error in cost

  then

  (i). closed loop system is stable if $w = 0$

  (ii). steady state error must be zero if response is stable for $w \neq 0$

- Robust MPC: use pre-stabilized predictions
  apply constraints for all possible future uncertainty

  then

  (i). constraint feasibility is guaranteed at all times if initially feasible

  (ii). closed loop system inherits the stability and convergence properties of unconstrained LQ optimal control (assuming nominal cost)

Overview of the course

1. Introduction and Motivation

   Basic MPC strategy; prediction models; input and state constraints; constraint handling: saturation, anti-windup, predictive control

2. Prediction and optimization

   Input/state prediction equations; unconstrained optimization. Infinite horizon cost; dual mode predictions. Incorporating constraints; quadratic programming.

3. Closed loop properties

   Lyapunov analysis based on predicted cost. Recursive feasibility; terminal constraints; the constraint checking horizon. Constrained LQ-optimal control.

4. Robustness to disturbances

   Setpoint tracking; MPC with integral action. Robustness to constant disturbances: prestabilized predictions and robust feasibility. Handling time-varying disturbances.