Model Predictive Control Examples Sheet

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Reading: Kouvaritakis & Cannon, Sections 2.1–2.6 and 3.1–3.3
or Maciejowski Chapters 2, 3, 6, 8

Prediction equations

1. A system with model

\[ x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k \]

\[ A = \begin{bmatrix} 1 & 0.1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

is to be controlled using an unconstrained predictive control law that minimizes the predicted performance cost

\[ J_k = \sum_{i=0}^{N-1} (y_{i|i|k}^2 + \lambda u_{i|i|k}^2) + y_{N|k}^2, \quad \lambda = 1. \]

(a). Show that the state predictions can be written in the form

\[ x_k = M x_k + C u_k, \quad u_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}, \quad x_k = \begin{bmatrix} x_{0|k} \\ \vdots \\ x_{N|k} \end{bmatrix} \]

and evaluate \( C \) and \( M \) for a horizon of \( N = 3 \).

(b). For \( N = 3 \), determine the matrices \( H, F \) and \( G \) in

\[ J_k = u_k^\top H u_k + 2 x_k^\top F u_k + x_k^\top G x_k. \]

(c). Give expressions for the derivatives \( \partial J/\partial u_{i|k} \) for \( i = 0, 1, 2 \). Hence verify that the gradient of \( J \) is \( \nabla_u J = 2Hu + 2Fx \).

2. (a). For the plant model and cost given in Question 1, show that the un-
constrained predictive control law for $N = 3$ is linear feedback:

$$u_k = L x_k, \quad L = - \begin{bmatrix} 0.1948 & 0.1168 \end{bmatrix}.$$ 

Hence show that the closed-loop system is unstable.

(b). Write some Matlab code to evaluate $\mathcal{M}$ and $\mathcal{C}$ for any given $N$, and hence determine $H$ and $F$, for any horizon length $N$. Show that the predictive control law does not stabilize the system if $N < 6$.

Infinite horizon cost and constraints

3. (a). Explain why the predictive control law of Question 1 necessarily stabilizes the system if the cost is minimized subject to $x_{N|k} = 0$.

(Hint: what is the infinite horizon cost when this constraint is used?)

(b). How would you modify the cost of Question 1 in order to achieve closed loop stability without including the constraint $x_{N|k} = 0$? Why would this be preferable?

4. A predictive controller minimizes the predicted performance index:

$$J_k = \sum_{i=0}^{\infty} (y^2_{i|k} + u^2_{i|k})$$

at each time-step $k$ subject to input constraints: $-1 \leq u_{i|k} \leq 2$ for all $i \geq 0$. The system output $y$ is related to the control input $u$ via

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = C x_k$$

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$ 

(a). Why is the MPC optimization performed repeatedly, at $k = 0, 1, 2 \ldots$, instead of just once, at $k = 0$?
(b). If the mode 2 feedback law is \( u_k = \begin{bmatrix} 2 & -1 \end{bmatrix} x_k \), show that
\[
J_k = \sum_{i=0}^{N-1} \left( y_{i|k}^2 + u_{i|k}^2 \right) + x_{N|k}^\top \begin{bmatrix} 13 & -1 \\ -1 & 2 \end{bmatrix} x_{N|k}
\]
where \( N \) is the length of the mode 1 prediction horizon.

(c). Show that the constraints
\[-1 \leq u_{i|k} \leq 2, \quad i = 0, 1, \ldots, N + 1\]
ensure that the predictions satisfy \(-1 \leq u_{i|k} \leq 2\) for all \( i \geq 0 \).

(d). Derive a bound on \( J_{k+1}^* - J_k^* \), where \( J_k^* \) is the optimal value of \( J_k \).
Hence show that \( \sum_{k=0}^{\infty} (y_k^2 + u_k^2) \leq J_0^* \) along trajectories of the closed loop system.

(e). Is the closed loop system stable? Explain your answer.

5. (a). Explain the function of terminal constraints in a model predictive control strategy for a system with input or state constraints. Define two principal properties that must be satisfied by a terminal constraint set.

(b). A discrete time system has the state space model
\[
x_{k+1} = Ax_k + Bu_k, \quad A = \begin{bmatrix} 0.3 & -0.9 \\ -0.4 & -2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}
\]
and constraints
\[|[x]_1 + [x]_2| \leq 1, \quad |[x]_1 - [x]_2| \leq 1, \quad x = \begin{bmatrix} [x]_1 \\ [x]_2 \end{bmatrix}\]

(i). If the terminal feedback law is \( u_k = K x_k, \quad K = \begin{bmatrix} 0.4 & 1.8 \end{bmatrix} \), show that the following set is a valid terminal constraint set
\[\{ x : |[x]_1 + [x]_2| \leq 1, \quad |[x]_1 - [x]_2| \leq 1 \}\.\]
(ii). Describe a procedure for determining the largest terminal constraint set for the case of a general feedback gain $K$.

(c). What are the main considerations that govern the choice of the prediction horizon $N$?

**Integral action and disturbances**

6. The vertical position $y$ of a machine tool positioning platform is controlled by a motor which applies a vertical force $F$ to the platform (Figure 1). The platform has mass $M$ and carries a variable load of mass $m$; the unloaded weight of the platform is balanced by a counter-weight. The force $F$ is proportional to the voltage $V$ applied to the motor, so that $F = K_V V$ where $K_V$ is a fixed gain.

![Figure 1. Machine tool and positioning platform](image)

Assuming $m$ is small enough that $M + m \approx M$, the unknown load constitutes a (constant) disturbance in the discrete-time model of the system for sampling interval $T$:

$$x_{k+1} = Ax_k + Bu_k + Dw, \quad e_k = Cx_k$$

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B = \frac{K_V}{2M} \begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \quad D = -\frac{g}{2M} \begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

where $e$ is the error in $y$ relative to a desired steady-state height $y^0$, and

$$x_k = \begin{bmatrix} y(kT) - y^0 \\ \dot{y}(kT) \end{bmatrix}, \quad u_k = V(kT), \quad w = m.$$

(a). For the model parameters $M = 10 \text{ kg}$, $K_V = 7 \text{ N V}^{-1}$, $T = 0.1 \text{ s}$, the
LQ-optimal feedback law with respect to the cost

\[ J_k = \sum_{i=0}^{\infty} \left( e_{k+i}^2 + \lambda u_{k+i}^2 \right), \quad \lambda = 10^{-4} \]

is \( u_k = K x_k, \ K = \begin{bmatrix} -66.0 & -19.4 \end{bmatrix} \). Determine the maximum steady state error \( y - y^0 \) with this controller if the mass of the load is limited to the range:

\[ m \leq 0.5 \text{ kg}. \]

(b). Explain how to modify the cost and model dynamics in order to obtain a stabilizing LQ-optimal controller giving zero steady-state error.

(c). The motor input voltage is subject to the constraints

\[ -1 \leq V \leq 1 \]

A predictive controller is to be designed based on the predicted cost:

\[ J_k = \sum_{i=0}^{\infty} \left( e_{i|k}^2 + v_{i|k}^2 + \lambda u_{i|k}^2 \right), \quad \lambda = 10^{-4} \]

where \( v_{i+1|k} = v_{i|k} + e_{i|k} \) is the prediction of the integrated error.

(i). For a predicted input sequence with \( N \) degrees of freedom, show that \( J_k \) can be re-written as

\[ J_k = \sum_{i=0}^{N-1} \left( e_{i|k}^2 + v_{i|k}^2 + \lambda u_{i|k}^2 \right) + \left\| \xi_{N|k} \right\|_P^2 \]

and define \( \xi \) and \( P \). What is the implied mode 2 feedback law?

(ii). Briefly explain how the constraints on \( V \) can be incorporated in a robust MPC strategy for this system (i.e. for all values of \( m \) in the range \( m \leq 0.5 \text{ kg} \)).

7. Assume that, for the given initial condition \( x(0) \), the optimization of \( J \) subject to the robust constraints determined in Question 6 is initially feasible.
Will the online optimization remain feasible at all future sampling times? What can be said about the steady-state value of $y$? Will the optimal value of the cost necessarily decrease monotonically, and what can be concluded about the convergence of the state $x_k$ to zero in closed-loop operation?

8. A production planning problem involves optimizing the quantity $u$ of stock manufactured in each week. The quantity $x$ of stock that remains unsold at the start of week $k + 1$ is given by

$$x_{k+1} = x_k + u_k - w_k, \quad k = 0, 1, \ldots$$

where the quantity $w_k$ that is sold in each week is unknown in advance but is expected to be equal to a known constant $\hat{w}$. Limits on storage and manufacturing capacities imply that $x$ and $u$ can only take values in the intervals

$$0 \leq x_k \leq X, \quad 0 \leq u_k \leq U.$$ 

The desired level of stock in storage is $x^*$, and the planned values $u_0|k, u_1|k, \ldots$ are to be optimized at time $k$ given a measurement of the value of $x_k$ by minimizing a cost

$$J_k = \sum_{i=0}^{\infty} e_i^2|k, e_i|k = x_i|k - x^*.$$ 

(a). What are the advantages of using a receding horizon control strategy in this application instead of an open-loop control sequence computed at $k = 0$?

(b). Assume that $w_k = \hat{w}$ for all $k = 0, 1, \ldots$

(i). Show that the unconstrained optimal control law is $u_k = \hat{w} - e_k$.

(ii). Show that, for a mode 1 horizon of $N$, the infinite horizon cost can be expressed

$$J_k = \sum_{i=0}^{N} e_i^2|k, e_i|k = \sum_{i=0}^{N} e_i^2|k.$$
and state the corresponding mode 2 feedback law.

(iii). Show that constraints are satisfied over an infinite horizon if \(0 \leq x_{i|k} \leq X\) and \(0 \leq u_{i|k} \leq U\) for \(0 \leq i \leq N - 1\), and

\[
\max \{0, \hat{w} + x^* - U\} \leq x_{N|k} \leq \min \{X, \hat{w} + x^*\}.
\]

What assumptions on \(\hat{w}, x^*, U\) and \(X\) are needed?

(c). Assume now that the future value of \(w\) is unknown and may take any value in an interval: \(0 \leq w_k \leq W\). Suggest how to express the planned sequence \(u_0|k, u_1|k, u_2|k\) in terms of the free variables in the receding horizon optimization problem, and justify your answer by determining the predictions \(e_1|k, e_2|k, e_3|k\).
Some answers

1. (b). \( H = \begin{bmatrix} 1.025 & 0.0075 & 0 \\ 0.0075 & 1.0025 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), \( F = \begin{bmatrix} 0.2 & 0.12 \\ 0.05 & 0.035 \\ 0 & 0 \end{bmatrix} \), \( G = \begin{bmatrix} 4 & 1.1 \\ 1.1 & 0.59 \end{bmatrix} \)

2. (a). Closed loop poles for \( N = 3 \): \( \text{eig}(A + BL) = 1.01, 1.93 \)

(b). \[
\begin{array}{c|cccc}
\text{eig}(A + BL) & 4 & 5 & 6 & 7 \\
\hline
1.03, 1.69 & 1.11 \pm 0.15i & 0.86 \pm 0.10i & 0.95, 0.58
\end{array}
\]

3. (b). In \( J_k \), replace \( y_{N|k}^2 \) with \( \|x_{N|k}\|_P^2 \), \( P = \begin{bmatrix} 22.46 & 4.098 \\ 4.098 & 12.79 \end{bmatrix} \)

4. (d). \( J_{k+1}^* - J_k^* \leq -(y_k^2 + u_k^2) \)

(e). \( x = 0 \) is locally asymptotically stable

6. (a). \( |y - y^0| \leq 0.0106 \text{ m in steady state} \)

(c). \( \xi \): augmented predicted state, \( \xi = \begin{bmatrix} x & v \end{bmatrix}^\top \), \( P \): the solution of \( P - \left( \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right)^\top P \left( \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right) = \begin{bmatrix} C^\top C & 0 \\ 0 & 1 \end{bmatrix} + \lambda K_\xi^\top K_\xi \), \( \xi \):

Mode 2 feedback law: \( u = K_\xi x \), e.g. LQ-optimal \( K_\xi = -\begin{bmatrix} 201.4 & 29.6 & 48.2 \end{bmatrix} \)

8. (c). \( u_{i|k} = \dot{w} - e_{i|k} + c_{i|k} \), where \( c_{i|k} = \) for \( i = 0, \ldots, N - 1 \) are decision variables, and \( c_{i|k} = 0 \) for \( i \geq N \)