Nonlinear Systems Examples Sheet

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Equilibrium points

1. (a). Find the equilibrium points of the system:

$$\dot{x} + x^3 = \sin^4 x$$

   (b). Rewrite the system model

$$\ddot{x} + (x-1)^2\dot{x}^5 + x^2 = \sin(\pi x/2)$$

   in terms of state variables $(x_1, x_2) = (x, \dot{x})$. Deduce that $\dot{x} = 0$ at an equilibrium point, and hence determine the values of x at equilibrium.

Lyapunov's direct method, invariant sets and linearization

2. The rotational motion of a drifting spacecraft is described by the dynamics

$$\dot{\omega}_x = a\omega_y\omega_z \qquad \dot{\omega}_y = -b\omega_x\omega_z \qquad \dot{\omega}_z = c\omega_x\omega_y$$

   where $\omega_x, \omega_y, \omega_z$ are angular velocities measured in a coordinate frame attached to the spacecraft (Fig. 1), and a, b, c are positive constants.

   (a). Determine the equilibrium points of this system.

   (b). Show that the equilibrium corresponding to zero rotation ($\omega_x = \omega_y = \omega_z = 0$) is stable.

   [*Hint:* Try using a storage function of the form $V = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$ with $ap - bq + cr = 0$. Is V positive definite? Does it satisfy $\dot{V} \leq 0$?]

   (c). Verify that the function

$$V = c\omega_y^2 + b\omega_z^2 + \left[2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)\right]^2$$

   satisfies $\dot{V} = 0$ along system trajectories, for any constant ω_0. What does this tell you about the stability of non-zero rotational motion?

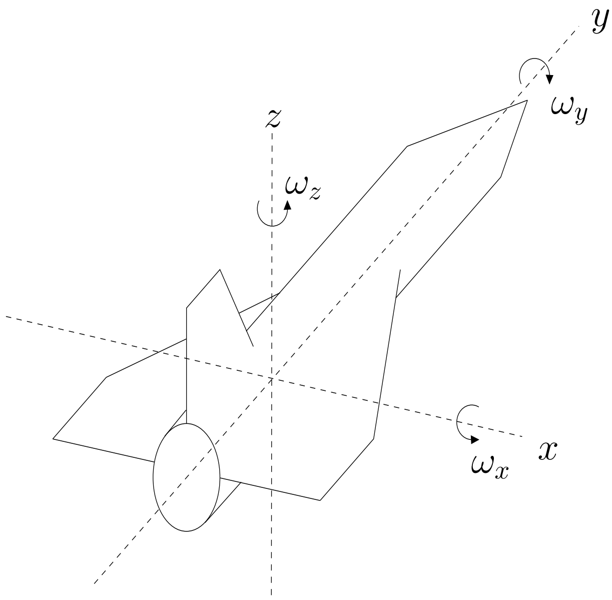

Figure 1: Rotating spacecraft.

3. (a). A first order system has model

$$\dot{x} + b(x) = 0 \qquad xb(x) > 0 \ \text{ for all } \ x \neq 0$$

where b is a continuous function. Show that $x = 0$ is a globally asymptotically stable equilibrium point.

(b). Find the equilibrium points of a second order system with model

$$\ddot{x} + b(\dot{x}) + c(x) = 0 \qquad \begin{array}{l} \dot{x}b(\dot{x}) > 0 \ \text{ for all } \ \dot{x} \neq 0 \\ xc(x) > 0 \ \text{ for all } \ x \neq 0 \end{array}$$

where b and c are continuous functions. By applying the invariant set theorem to the function

$$V(x) = \tfrac{1}{2}\dot{x}^2 + \int_0^x c(s)ds$$

show that $(x, \dot{x}) = (0, 0)$ is asymptotically stable. What extra conditions are needed to show global asymptotic stability using V?

4. Consider the second order system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2(x_1 - 1)^2 - x_1(x_1^2 - 1). \end{aligned}$$

(a). Determine the equilibrium points of the system.

(b). Use the function

$$V(x_1, x_2) = \tfrac{1}{4}x_1^2(x_1^2 - 2) + \tfrac{1}{2}x_2^2,$$

to show that every state trajectory tends to an equilibrium point.

(c). Show that the equilibrium point at $(x_1, x_2) = (0, 0)$ is unstable using Lyapunov's linearization method.

(d). Use the function $U(x_1, x_2) = V(x_1, x_2) + \frac{1}{4}$ to show that the other two equilibrium points are stable.

5. A system described by the nonlinear model

$$\dot{x} = Ax + (B + x)u \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is to be controlled using linear state feedback $u = -Kx$ with $K = [1 \;\; 1]$.

(a). Find the matrix Q satisfying

$$(A - BK)^T P + P(A - BK) = -Q \qquad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and verify that P and Q are positive definite matrices. Use this result to determine whether the closed loop system is stable at the equilibrium point $x = 0$.

(b). Show that the storage function $V = x^T P x$ satisfies

$$\dot{V} \leq -x^T Q x (1 - 2|Kx|)$$

along trajectories of the closed loop system.

(c). Use the bound on $\dot{V}$ given in part (b) to determine a region of state space within which $\dot{V}$ is negative definite. Show that

$$\Omega = \{x : x^T P x \leq \alpha\}$$

defines a region of attraction of $x = 0$ whenever α is less than some maximum value (there is no need to determine this maximum value).

Linear and passive systems

6. Show that the real parts of the eigenvalues of A satisfy $\mathrm{Re}\lambda(A) < -\mu$ if there exist symmetric positive definite matrices P and Q satisfying $A^T P + PA + 2\mu P = -Q$ for $\mu > 0$.

7. The nonlinear LCR circuit shown in Figure 2 is described by the equations:

$$\dot{x}_1 = x_2/L$$
$$\dot{x}_2 + x_1/C + x_2 R_1/L = e$$

where $x_1(t)$ is the charge on the capacitor and $x_2(t)$ is the magnetic flux in the inductor. Capacitance C depends on x_1, inductance L depends on x_2, and the resistance R_1 is time-varying, with $C(x_1) > 0$ for all x_1, $L(x_2) > 0$ for all x_2, and $R_1(t) > 0$ for all t.

(a). Use the function:

$$V_1(x_1, x_2) = \int_0^{x_2} \frac{x}{L(x)}\, dx \; + \; \int_0^{x_1} \frac{x}{C(x)}\, dx$$

to show that the system with $e(t)$ as input $\dot{x}_1(t)$ as output is passive.

(b). For the circuit in Figure 3 with switch S closed, find a function V satisfying

$$V \geq 0, \qquad \dot{V} = ie - \frac{R_1}{L^2(x_2)} x_2^2 - \frac{R_2}{L^2(x_4)} x_4^2$$

where x_2, x_4 are the fluxes in the two inductors. If $R_2(t) > 0$ for all t, what does this imply about the stability of the circuit with S open?

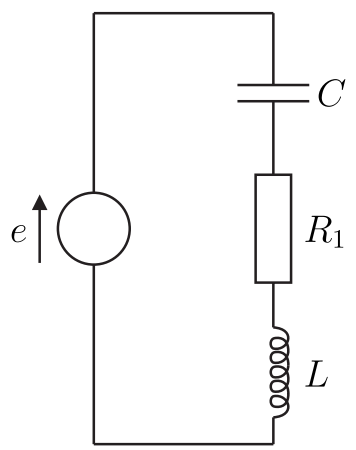

Figure 2

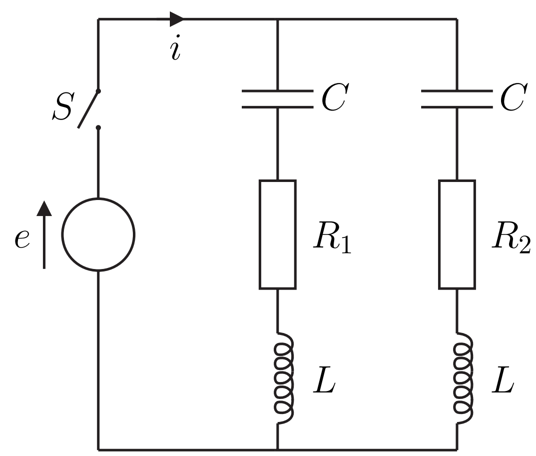

Figure 3

8. A linear system with input u, output y and stable open-loop transfer function $G(s)$ is to be controlled via feedback $u = -\phi(y)$, where ϕ is a static nonlinearity. For all ω, $G(j\omega)$ lies within the bounds:

$$-1 < \mathrm{Re}\big[G(j\omega)\big] < 2, \qquad -2 < \mathrm{Im}\big[G(j\omega)\big] < 2.$$

(a). Show that the closed-loop system is asymptotically stable for any function ϕ belonging to the sector $[0, 1]$ or $[-\frac{1}{3}, \frac{1}{2}]$.

(b). Does this imply that the closed-loop system will be stable for all ϕ in the sector $[-\frac{1}{3}, 1]$? Explain your answer.

Some answers

1. (a). $x = 0$ (b). $(x, \dot{x}) = (0,0), (1,0)$
2. (a). Any two of $\omega_x, \omega_y, \omega_z$ must be zero.
4. (a). $(x_1, x_2) = (0,0), (1,0), (-1,0)$
5. (a). $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
7. (a). $V = \int_0^{x_2} \frac{x}{L(x)}\, dx + \int_0^{x_4} \frac{x}{L(x)}\, dx + \int_0^{x_1} \frac{x}{C(x)}\, dx + \int_0^{x_3} \frac{x}{C(x)}\, dx$

   (b). The system is locally asymptotically stable (or globally asymptotically stable if $\int_0^{x_2} \frac{x}{L(x)}\, dx \to \infty$ as $|x_2| \to \infty$ and $\int_0^{x_1} \frac{x}{C(x)}\, dx \to \infty$ as $|x_1| \to \infty$).