C21 Nonlinear Systems

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4 lectures

Hilary Term 2023

Department of Engineering Science
eng.ox.ac.uk/control

Lecture 1

Introduction and Concepts of Stability

Organisation

- 4 lectures – LR2, weeks 1 & 2
Monday at 15.00 & Friday at 12.00
recordings available on Canvas
- Examples class – LR3, week 3
Friday at 14.00, 16.00 or 17.00
sign up on Canvas

Course outline

1. Types of stability
2. Linearization
3. Lyapunov's direct method
4. Regions of attraction
5. Linear systems and passive systems

Books

- ▷ J.-J. Slotine & W. Li *Applied Nonlinear Control*, Prentice-Hall 1991.

  Chapters 3 & 4

- ▷ H.K. Khalil *Nonlinear Systems*, Prentice-Hall 1996.

  Chapters 1, 3, 4, 10 and 11

- ▷ M. Vidyasagar *Nonlinear Systems Analysis*, Prentice-Hall 1993.

  Chapter 5

- ▷ K.J. Astrom and R.M. Murray *Feedback Systems: an introduction for scientists and engineers*, Princeton University Press, 2008.

  Chapter 4

Why use nonlinear control?

- Real systems are nonlinear
  - friction, non-ideal components
  - actuator saturation
  - sensor nonlinearity
- Analysis via linearization
  - accuracy of approximation?
  - conservative?
- Account for nonlinearities in high performance applications
  - Robotics, Aerospace, Petrochemical industries, Process control, Power generation ...
- Account for nonlinearities if linear models inadequate
  - large operating region
  - model properties change at linearization point

Linear systems reminder - 1/2

Linear system free response

$$\dot{x} = Ax$$

Eigen-decomposition: $Av_i = v_i \lambda_i$

let $V = \begin{bmatrix} v_1, \dots, v_n \end{bmatrix}$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

then $A = V \Lambda V^{-1}$ (if V^{-1} exists)

$$\Rightarrow \quad \dot{z} = \Lambda z, \qquad z = V^{-1}x$$

$$z(t) = e^{\Lambda t} z(0)$$

$$\Rightarrow \quad x(t) = V e^{\Lambda t} V^{-1} x(0)$$

$$= e^{At} x(0)$$

System is stable if $\text{Re}(\lambda_i) < 0 \ \forall i$

Linear vs nonlinear system properties 1/2

Free response

Linear system

$$\dot{x} = Ax$$

- Unique equilibrium point:
$$Ax = 0 \iff x = 0$$
- Stability independent of initial conditions

Nonlinear system

$$\dot{x} = f(x)$$

- Multiple equilibrium points
$$f(x) = 0$$
- Stability dependent on initial conditions

Linear systems reminder 2/2

Linear system free response

$$\dot{x} = Ax$$

Eigen-decomposition: $Av_i = v_i \lambda_i$

Let $V = [v_1, \ldots, v_n]$

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then $A = V \Lambda V^{-1}$ (if V^{-1} exists)

$$\Rightarrow \quad \dot{z} = \Lambda z, \quad z = V^{-1} x$$

$$z(t) = e^{\Lambda t} z(0)$$

$$\Rightarrow \quad x(t) = V e^{\Lambda t} V^{-1} x(0) = e^{At} x(0)$$

System is stable if $\text{Re}(\lambda_i) < 0$

Forced response

$$\dot{x} = Ax + Bu$$

$$\Rightarrow \quad x(t) = \int_0^t e^{A(t-h)} Bu(h)\, dh + e^{At} x(0)$$

If $\text{Re}(\lambda_i) < 0$, then the system is input-to-state stable:

$$\|x(t)\| \leq \|e^{At} x(0)\| + \gamma \sup_{t \geq 0} \|u(t)\|$$

$$\gamma = \|B\| \int_0^\infty \|e^{At}\| dt$$

Frequency response

$$\dot{x} = Ax + Bu$$

$$u = U(\omega) e^{j\omega t} \quad \Rightarrow \quad x = X(\omega) e^{j\omega t}$$

$$\Rightarrow \quad X(\omega) = (j\omega I - A)^{-1} BU(\omega)$$

Linear vs nonlinear system properties 2/2

Forced response

Linear system

$$\dot{x} = Ax + Bu$$

- $\|u\|$ finite $\Rightarrow$ $\|x\|$ finite if open-loop stable
- Frequency response:
  $u = U \sin \omega t \;\; \Rightarrow \;\; x = X \sin(\omega t + \phi)$
- Superposition:
  $u = u_1 + u_2 \;\; \Rightarrow x = x_1 + x_2$

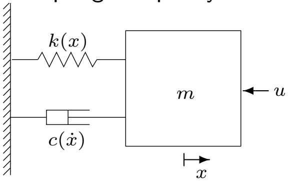
Nonlinear system

$$\dot{x} = f(x, u)$$

- $\|u\|$ finite $\not\Rightarrow$ $\|x\|$ finite
- No frequency response
  $u = U \sin \omega t \;\; \not\Rightarrow \;\; x$ sinusoidal
- No linear superposition
  $u = u_1 + u_2 \;\; \not\Rightarrow x = x_1 + x_2$

Example: step response

Mass-spring-damper system

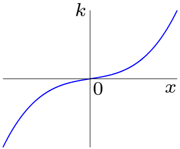


Equation of motion:

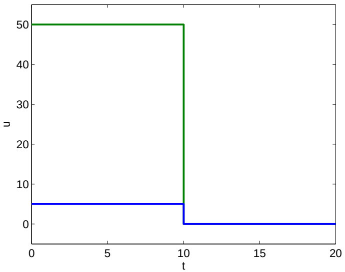
$$m\ddot{x} + c(\dot{x}) + k(x) = u$$

$c(\dot{x}) = \dot{x}$

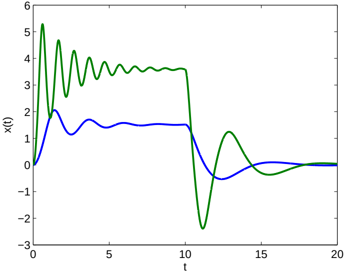
$k(x)$ nonlinear:



Input $u(t)$



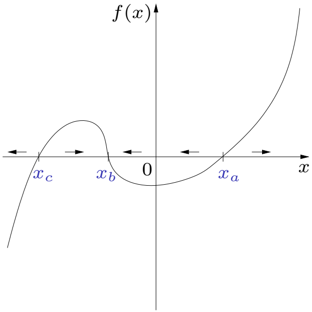
Response $x(t)$



apparent damping ratio depends on size of input step

Example: multiple equilibria

First order system: $\dot{x} = f(x)$

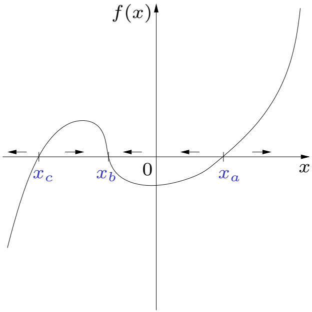


$x > x_a$	$\Longrightarrow$	$f(x) > 0$	$\Longrightarrow$	$x(t)$ increases
$x_b < x < x_a$	$\Longrightarrow$	$f(x) < 0$	$\Longrightarrow$	$x(t)$ decreases
$x_c < x < x_b$	$\Longrightarrow$	$f(x) > 0$	$\Longrightarrow$	$x(t)$ increases
$x < x_c$	$\Longrightarrow$	$f(x) < 0$	$\Longrightarrow$	$x(t)$ decreases

- x_a, x_c are unstable equilibrium points
- x_b is a stable equilibrium point

Example: multiple equilibria

First order system: $\dot{x} = f(x)$



$$
\begin{array}{llll}
x > x_a & \implies f(x) > 0 & \implies & x(t) \text{ increases} \\
x_b < x < x_a & \implies f(x) < 0 & \implies & x(t) \text{ decreases} \\
x_c < x < x_b & \implies f(x) > 0 & \implies & x(t) \text{ increases} \\
x < x_c & \implies f(x) < 0 & \implies & x(t) \text{ decreases}
\end{array}
$$

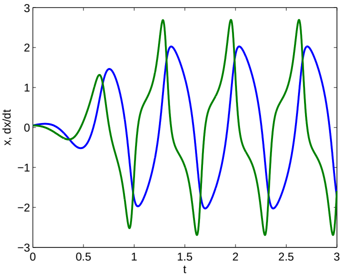
- x_a, x_c are unstable equilibrium points
- x_b is a stable equilibrium point

Example: limit cycle

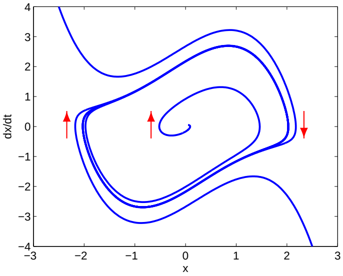
Van der Pol oscillator:

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

- Response $x(t)$ tends to a limit cycle (= trajectory forming a closed curve)
- Amplitude independent of initial conditions



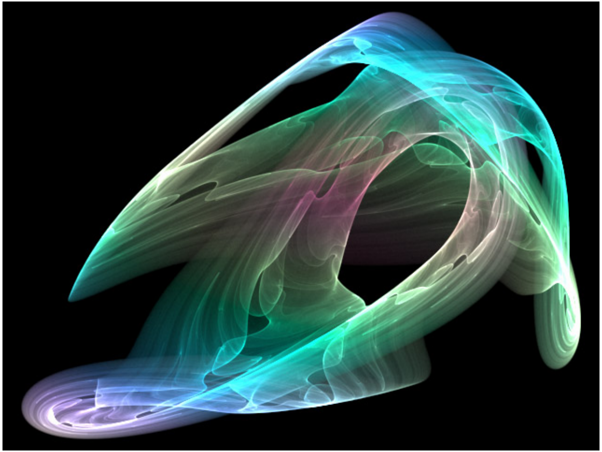
Response with $x(0) = 0.05$, $\dot{x}(0) = 0.05$



State trajectories $\big(x(t), \dot{x}(t)\big)$

Example: chaotic behaviour

Strange attractor



Example: chaotic behaviour

Lorenz attractor

- Simplified model of atmospheric convection:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

- State variables

$x(t)$: fluid velocity

$y(t)$: difference in temperature of acsending and descending fluid

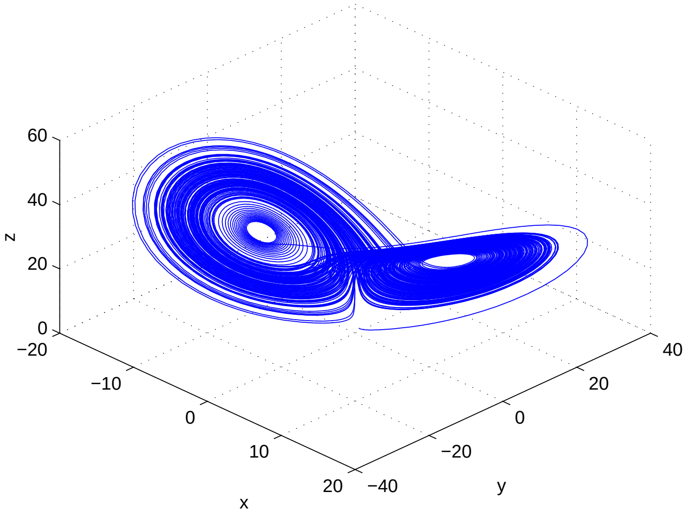
$z(t)$: characterizes distortion of vertical temperature profile

- Parameters $\sigma = 10$, $\beta = 8/3$, $\rho =$ variable

Example: chaotic behaviour

Lorenz attractor

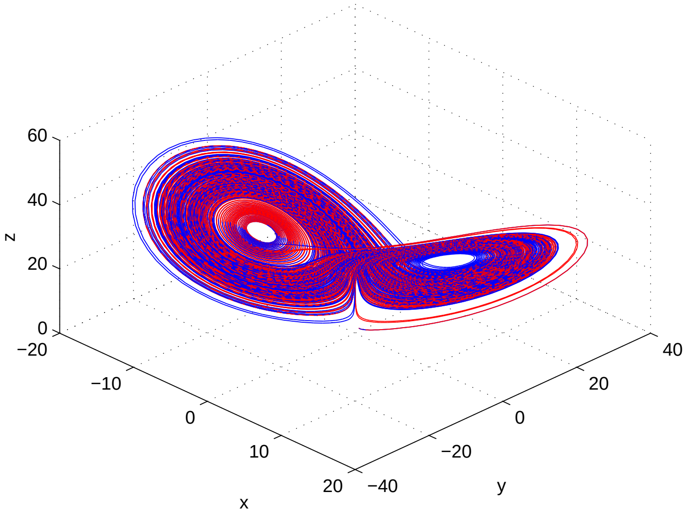
$\rho = 28 \implies$ "strange attractor":



Example: chaotic behaviour

Lorenz attractor

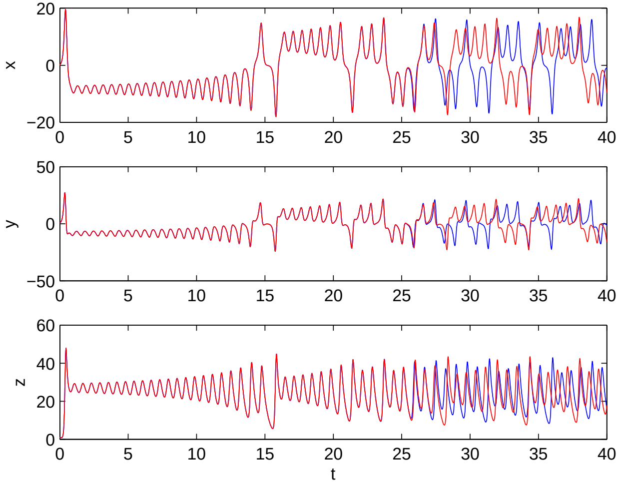
sensitivity to initial conditions



Example: chaotic behaviour

Lorenz attractor

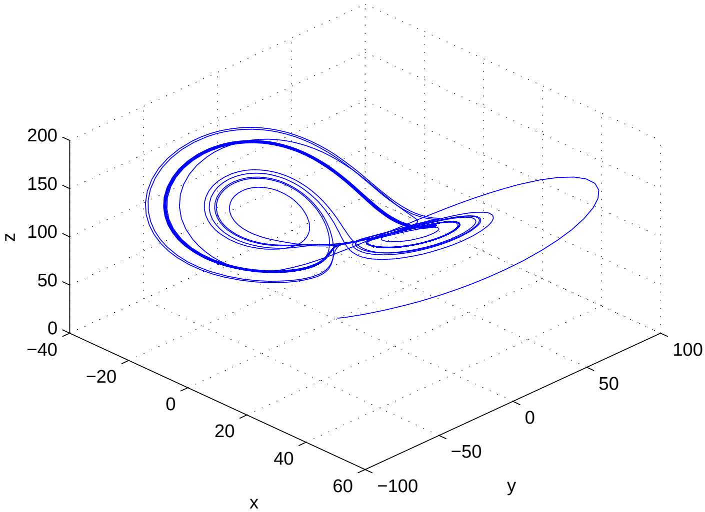
sensitivity to initial conditions — blue: $(x, y, z) = (0, 1, 1.05)$
red: $(x, y, z) = (0, 1, 1.050001)$



Example: chaotic behaviour

Lorenz attractor

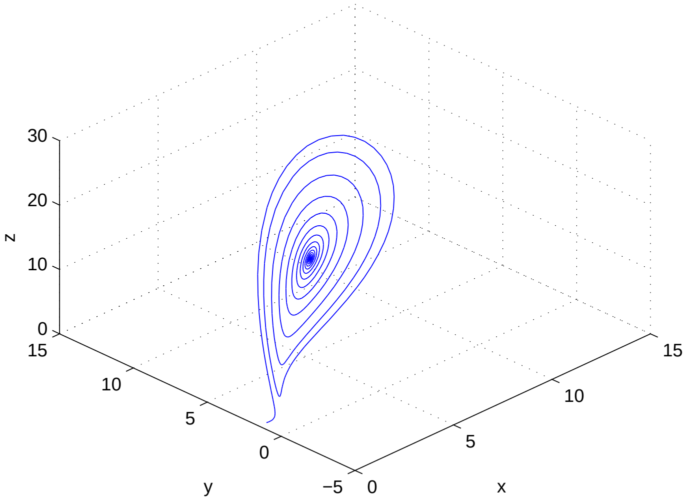
$\rho = 99.96 \implies$ limit cycle:



Example: chaotic behaviour

Lorenz attractor

$\rho = 14 \implies$ convergence to a stable equilibrium:



State space equations

A continuous-time nonlinear system

$$\dot{x} = f(x, u, t) \qquad \begin{array}{rcl} x & : & \text{state} \\ u & : & \text{input} \end{array}$$

e.g. nth order differential equation:

$$\frac{d^n y}{dt^n} = h\Big(y, \ldots, \frac{d^{n-1} y}{dt^{n-1}}, u, t\Big)$$

has state vector (one possible choice)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ d^{n-1}y/dt^{n-1} \end{bmatrix}$$

and state space dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ h\big(x_1, x_2, \ldots, x_n, u, t\big) \end{bmatrix} = f(x, u, t)$$

Equilibrium points

x^* is an equilibrium point of system $\dot{x} = f(x)$ if (and only if):

$$x(0) = x^* \quad \text{implies} \quad x(t) = x^* \quad \forall t > 0$$

i.e. $f(x^*) = 0$

- ⋆ Consider local stability of individual equilibrium points
- ⋆ Convention: define f so that $x = 0$ is equilibrium point of interest
- ⋆ Autonomous system: $\dot{x} = f(x) \quad \Longrightarrow \quad x^* = \text{constant}$

Examples:

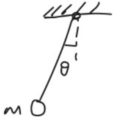
(1). $\ddot{\theta} + \alpha\dot{\theta}^2 + \beta \sin\theta = 0$ (pendulum with damping)

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad x^* = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \; n = 0, \pm 1$$

(2). $\ddot{y} + (y-1)^2\dot{y} + y - \sin(\pi y/2) = 0$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Equilibrium points



① $\ddot{\theta} + \alpha\dot{\theta}^2 + mg\sin\theta = 0$

STATE: $x = \begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} \quad \dot{x} = \begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ -\alpha\dot{\theta}^2 - mg\sin\theta\end{bmatrix}$

EQM: $\dot{x} = 0 \Rightarrow \left.\begin{array}{l}\dot{\theta} = 0\\ \sin\theta = 0\end{array}\right\} x^* = \begin{bmatrix}n\pi\\ 0\end{bmatrix} \quad n = 0, \pm 1, \ldots$

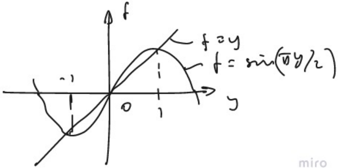
② $\ddot{y} + (y-1)^2\dot{y} + y - \sin(\pi y/2) = 0$

STATE: $x = \begin{bmatrix}y\\ \dot{y}\end{bmatrix}, \quad \dot{x} = \begin{bmatrix}\dot{y}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}\dot{y}\\ -(y-1)^2\dot{y} - y + \sin(\pi y/2)\end{bmatrix}$

EQM: $\dot{x} = 0 \Rightarrow \dot{y} = 0$

$y - \sin(\pi y/2) = 0$

$\Rightarrow x^* = \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\pm 1\\0\end{pmatrix}$



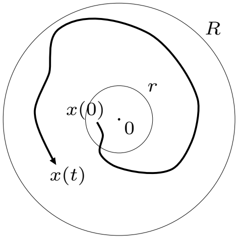
Stability definition

An equilibrium point $x = 0$ is stable iff:

$$\max_t \|x(t)\| \text{ can be made arbitrarily small}$$
$$\text{by making } \|x(0)\| \text{ small enough}$$

$$\Updownarrow$$

for any $R > 0$, there exists $r > 0$ so that
$$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > 0$$



- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
- No guarantee of convergence to the equilibrium point

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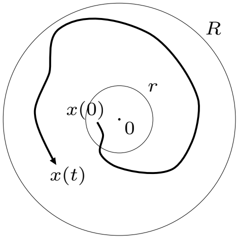


- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
- No guarantee of convergence to the equilibrium point

Asymptotic stability definition

An equilibrium point $x = 0$ is asymptotically stable iff:

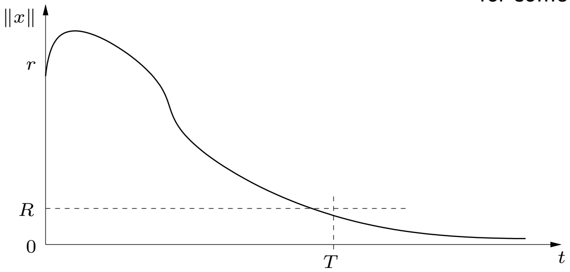
(i). $x = 0$ is stable
(ii). $\|x(0)\| < r \implies \|x(t)\| \to 0$ as $t \to \infty$

(ii) is equivalent to:

for any $R > 0$,

$$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > T$$

for some r, T

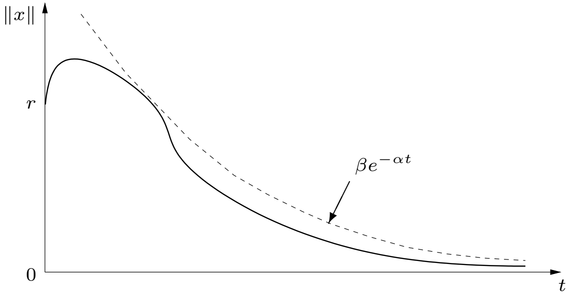


Exponential stability definition

An equilibrium point $x = 0$ is exponentially stable iff:

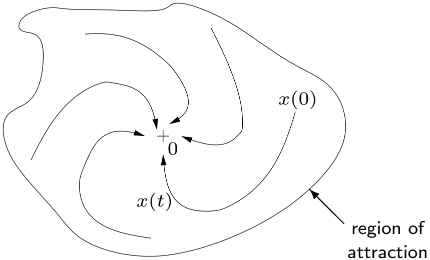
$$\|x(0)\| < r \implies \|x(t)\| \le \beta e^{-\alpha t} \quad \forall t > 0$$

exponential stability is a special case of asymptotic stability



Region of attraction

The region of attraction of $x = 0$ is the set of all initial conditions $x(0)$ for which $x(t) \to 0$ as $t \to \infty$



- Every asymptotically stable equilibrium point has a region of attraction
- $r = \infty \quad \Longrightarrow \quad$ entire state space is a region of attraction
  $\Longrightarrow \quad x = 0$ is globally asymptotically stable
- Are stable linear systems asymptotically stable?

Summary

- Nonlinear state space equations: $\dot{x} = f(x, u)$
  x = state vector, u = control input

- Equilibrium points: x^* is an equilibrium point
  of $\dot{x} = f(x)$ if $f(x^*) = 0$

- Stable equilibrium point: x^* is stable if state trajectories starting
  close to x^* remain near x^* at all times

- Asymptotically stable equilibrium point: x^* must be stable and state trajectories starting near x^* must tend to x^* asympotically

- Region of attraction: the set of initial conditions from which state trajectories converge asymptotically to equilibrium x^*

Lecture 2

Linearization and Lyapunov's direct method

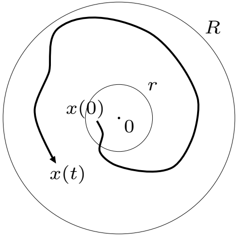
Linearization and Lyapunov's direct method

- Review of stability definitions
- Linearization method
- Direct method for stability
- Direct method for asymptotic stability
- Linearization method revisited

Review of stability definitions

System: $\dot{x} = f(x)$

- ⋆ unforced system (i.e. closed-loop)
- ⋆ consider stability of individual equilibrium points



0 is a stable equilibrium if:

$$\|x(0)\| \le r \implies \|x(t)\| \le R$$

for any $R > 0$

0 is asymptotically stable if:

$$\|x(0)\| \le r \implies \|x(t)\| \to 0$$

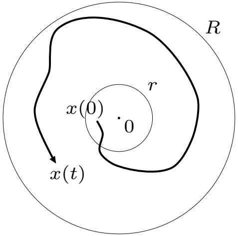
as $t \to \infty$

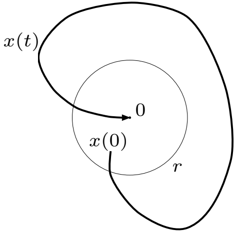
Stability	→	local property
Asymptotic stability	→	global if $r = \infty$ allowed

Review of stability definitions

System: $\dot{x} = f(x)$

- ⋆ unforced system (i.e. closed-loop)
- ⋆ consider stability of individual equilibrium points





0 is a stable equilibrium if:

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for any $R > 0$

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as $t \to \infty$

Stability	$\rightarrow$	local property
Asymptotic stability	$\rightarrow$	global if $r = \infty$ allowed

Review of stability definitions

NON-AUTONOMOUS SYSTEMS : $\dot{x} = f(x, t)$ [non-examinable!]

* STABLE EQUILIBRIUM : $\forall R > 0 \;\exists r$ SO THAT

$$\|x(t_0)\| \leq r \Rightarrow \|x(t)\| \leq R \quad \forall t \geq t_0 .$$

r INDEPENDENT OF $t_0 \Rightarrow x = 0$ IS UNIFORMLY STABLE

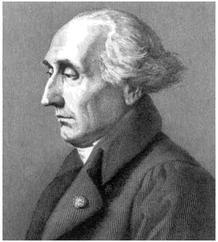
* ASYMPTOTICALLY STABLE EQM : $\exists r$ SO THAT, $\forall R > 0 \;\exists T$ SO THAT

$$\|x(t_0)\| \leq r \Rightarrow \|x(t)\| \leq R \quad \forall t \geq t_0 + T .$$

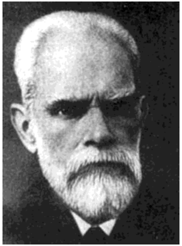
r, T INDEPENDENT OF $t_0 \Rightarrow x = 0$ IS UNIFORMLY ASYMPTOTICALLY STABLE

Historical development of Stability Theory

- Potential energy in conservative mechanics (Lagrange 1788):
  An equilibrium point of a conservative system is stable if it corresponds to a minimum of the potential energy stored in the system
- Energy storage analogy for general ODEs (Lyapunov 1892)
- Invariant sets (Lefschetz, La Salle 1960s)



J-L. Lagrange 1736-1813



A. M. Lyapunov 1857-1918

S. Lefschetz 1884-1972

Lyapunov's linearization method

- Determine stability of equilibrium at $x = 0$ by analyzing the stability of the linearized system at $x = 0$.

- Jacobian linearization:

$$\begin{aligned} \dot{x} &= f(x) && \text{original nonlinear dynamics} \\ &= f(0) + \frac{\partial f}{\partial x}\Big|_{x=0} x + R_1(x) && \text{Taylor's series expansion} \\ &\approx Ax && \text{since } f(0) = 0 \end{aligned}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \text{Jacobian matrix}$$

$$R_1(x) \to 0 \ \text{ as } x \to 0 \qquad \text{remainder}$$

Taylor's Theorem reminder

- IF $f(x)$ IS DIFFERENTIABLE AT $x=0$, THEN

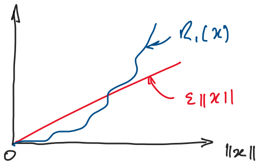
$$f(x) = f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + R_1(x)$$

WHERE $R_1(x) \to 0$ AS $x \to 0$

- MORE PRECISELY : $\lim_{x \to 0} \frac{R_1(x)}{\|x\|} = 0$

WHICH IMPLIES THAT, FOR ANY $\varepsilon > 0$, THERE EXISTS AN $r > 0$ SUCH THAT

$$\|R_1(x)\| \leq \varepsilon \|x\| \quad \text{if} \quad \|x\| \leq r$$



Lyapunov's linearization method

Conditions on A for stability of original nonlinear system at $x = 0$:

stability of linearization	stability of nonlinear system at $x = 0$
$\text{Re}\big(\lambda(A)\big) < 0$	asymptotically stable (locally)
$\max \text{Re}\big(\lambda(A)\big) = 0$	stable or unstable
$\max \text{Re}\big(\lambda(A)\big) > 0$	unstable

Lyapunov's linearization method: examples

$\ast \quad \dot{x} = x\cos x \quad \Rightarrow \quad \dot{x} \simeq \left[\cos x - x\sin x\right]_{x=0} \cdot x = x$

or $\quad \dot{x} = x\left(1 - \frac{x^2}{2} + \dots\right) \approx x$

LINEARISATION: $\dot{x} = x \quad \Rightarrow \quad \lambda = 1 \quad \Rightarrow \quad x = 0$ IS AN UNSTABLE EQM

$\ast \quad \ddot{x} + \dot{x}e^{x} = 0 \quad \Rightarrow \quad \ddot{x} + \dot{x}\left(1 + x + \frac{x^2}{2} + \dots\right) = 0$

$\Rightarrow \quad \ddot{x} + \dot{x} + \text{h.o.t.} = 0$

$\Rightarrow \quad \ddot{x} + \dot{x} \approx 0$

LINEARISATION: $\ddot{x} + \dot{x} = 0 \quad \Rightarrow \quad \lambda = -1, 0 \quad \Rightarrow$ INCONCLUSIVE

Lyapunov's linearization method

- Linearization may not provide enough information:

$$\text{(stable)} \qquad \dot{x} = -x^3 \quad \xrightarrow{\text{linearize}} \quad \dot{x} = 0 \qquad \text{(indeterminate)}$$

$$\text{(unstable)} \qquad \dot{x} = x^3 \quad \xrightarrow{\text{linearize}} \quad \dot{x} = 0 \qquad \text{(indeterminate)}$$

↑ higher order terms determine stability

- Why does linear control work?
  1. Linearize the model:

$$\dot{x} = f(x,u)$$

$$\approx Ax + Bu, \qquad A = \frac{\partial f}{\partial x}(0,0), \ B = \frac{\partial f}{\partial u}(0,0)$$

  2. Design a linear feedback controller using the linearized model:

$$u = -Kx, \quad \max \mathrm{Re}\big(\lambda(A - BK)\big) < 0$$

closed-loop linear model strictly stable

nonlinear system $\dot{x} = f(x, -Kx)$ is locally asymptotically stable at $x = 0$

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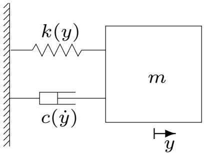
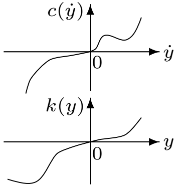
Lyapunov's direct method: mass-spring-damper example





Equation of motion: $m\ddot y + c(\dot y) + k(y) = 0$

Stored energy: $V = \text{K.E.} + \text{P.E.}$ $\begin{cases} \text{K.E.} = \frac{1}{2}m\dot y^2 \\ \text{P.E.} = \int_0^y k(y)\,dy \end{cases}$

Rate of energy dissipation $\dot V = \frac{1}{2}m\ddot y \frac{d}{d\dot y}\dot y^2 + \dot y \frac{d}{dy}\left[\int_0^y k(y)\,dy\right]$

$= m\ddot y \dot y + \dot y k(y)$

but $m\ddot y + k(y) = -c(\dot y)$, so $\dot V = -c(\dot y)\dot y$

≤ 0 $\leftarrow$ since $\operatorname{sign}\big(c(\dot y)\big) = \operatorname{sign}(\dot y)$

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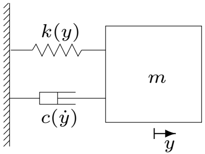

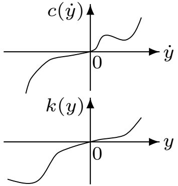

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$$\begin{aligned} \dot V &= \tfrac{1}{2}m\ddot y \frac{d}{d\dot y}\dot y^2 + \dot y \frac{d}{dy}\left[\int_0^y k(y)\,dy\right] \\ &= m\ddot y\dot y + \dot y k(y) \end{aligned}$$

but $m\ddot y + k(y) = -c(\dot y)$, so

$$\begin{aligned} \dot V &= -c(\dot y)\dot y \\ &\le 0 \qquad \leftarrow \text{since } \operatorname{sign}\big(c(\dot y)\big) = \operatorname{sign}(\dot y) \end{aligned}$$

Mass-spring-damper example contd.

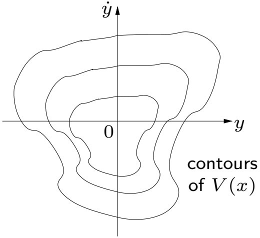
- System state: e.g. $x = [y \;\; \dot{y}]^T$
- $\dot{V}(x) \leq 0$ implies that $x = 0$ is stable

  $\uparrow$

  $V(x(t))$ must decrease over time

  but

  $V(x)$ increases with increasing $\|x\|$



- Formal argument:

  for any given $R > 0$:

  $\|x\| < R$ whenever $V(x) < \overline{V}$ for some $\overline{V}$

  and $V(x) < \overline{V}$ whenever $\|x\| < r$ for some r

  $\therefore\ \|x(0)\| < r \implies V(x(0)) < \overline{V}$

  $\implies V(x(t)) < \overline{V}$ for all $t > 0$

  $\implies \|x(t)\| < R$ for all $t > 0$

Mass-spring-damper example contd.

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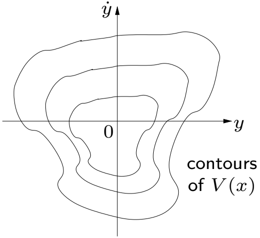

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$$\begin{aligned} \therefore \|x(0)\| < r \quad &\implies \quad V\big(x(0)\big) < \overline{V} \\ &\implies \quad V\big(x(t)\big) < \overline{V} \quad \text{for all } t > 0 \\ &\implies \quad \|x(t)\| < R \quad \text{for all } t > 0 \end{aligned}$$

Positive definite functions

- What if $V(x)$ is not monotonically increasing in $\|x\|$?
- Same arguments apply if $V(x)$ is continuous and positive definite, i.e.

$$\begin{array}{lll} \text{(i).} & V(0) = 0 & \\ \text{(ii).} & V(x) > 0 & \text{for all } x \neq 0 \end{array}$$

for any given $\overline{V} > 0$,
can always find r so that

$V(x) < \overline{V}$ whenever $\|x\| < r$

$V(x) \geq \alpha(\|x\|)$ for some continuous
and strictly increasing function $\alpha(\cdot)$, So
$\|x\| < \alpha^{-1}(\overline{V})$ whenever $V(x) < \overline{V}$

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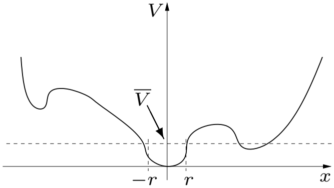

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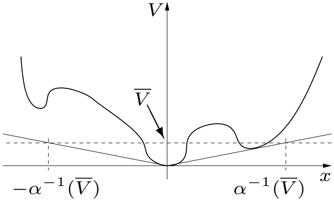

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Lyapunov stability theorem

If there exists a continuous function $V(x)$ such that

$$\begin{array}{l} V(x) \text{ is positive definite} \\ \dot{V}(x) \leq 0 \end{array}$$

then $x = 0$ is stable.

To show that this implies $\|x(t)\| < R$ for all $t > 0$ whenever $\|x(0)\| < r$ for any R and some r:

1. choose $\overline{V}$ as the minimum of $V(x)$ subject to $\|x\| = R$
2. find r so that $V(x) < \overline{V}$ whenever $\|x\| < r$
3. then $\dot{V}(x) \leq 0$ ensures that

$$\begin{array}{l} V\big(x(t)\big) < \overline{V} \quad \forall t > 0 \quad \text{if } \|x(0)\| < r \\ \therefore \|x(t)\| < R \quad \forall t > 0 \end{array}$$

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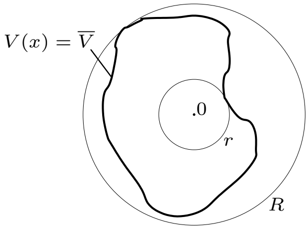

Lyapunov stability theorem

- Lyapunov's direct method also applies if $V(x)$ is locally positive definite, i.e. if

$$\begin{array}{lll} \text{(i).} & V(0) = 0 & \\ \text{(ii).} & V(x) > 0 & \text{for } x \neq 0 \text{ and } \|x\| < R_0 \end{array}$$

  then $x = 0$ is stable if $\dot{V}(x) \leq 0$ whenever $\|x\| < R_0$.

- Apply the theorem without determining R, r:
  we only need to find p.d. $V(x)$ satisfying $\dot{V}(x) \leq 0$.

- Examples

  (i). $\dot{x} = -a(t)x, \quad a(t) > 0$

$$V = \tfrac{1}{2}x^2 \quad \Longrightarrow \quad \begin{aligned} \dot{V} &= x\dot{x} \\ &= -a(t)x^2 \leq 0 \end{aligned}$$

  (ii). $\dot{x} = -a(x), \quad \mathrm{sign}\big(a(x)\big) = \mathrm{sign}(x)$

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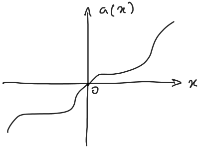
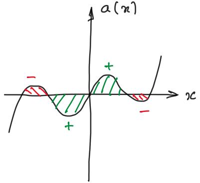


Lyapunov stability theorem

- More examples

(iii). $\dot{x} = -a(x), \quad \int_0^x a(x)\,dx > 0$



$$V = \int_0^x a(x)\,dx \quad \Longrightarrow \quad \dot{V} = a(x)\dot{x}$$

$$= -a^2(x) \le 0$$

(iv). $\ddot{\theta} + \sin\theta = 0$

$$V = \tfrac{1}{2}\dot{\theta}^2 + \int_0^\theta \sin\theta\,d\theta \quad \Longrightarrow \quad \dot{V} = \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta$$

$$= 0$$

Asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}$$

then $x = 0$ is locally asymptotically stable.

($\dot{V}$ negative definite $\iff$ $-\dot{V}$ positive definite)

Asymptotic convergence $x(t) \to 0$ as $t \to \infty$ can be shown by contradiction:

if $\|x(t)\| > R'$ for all $t \geq 0$, then

$$\left.\begin{array}{l} \dot{V}(x) < -W \\ \\ V(x) \geq \underline{V} \end{array}\right\} \quad \text{for all } t \geq 0$$

↑ contradiction

Asymptotic stability theorem

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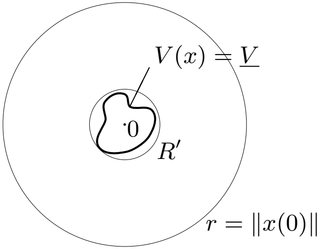

Linearization method and asymptotic stability

- Asymptotic stability result also applies if $\dot{V}(x)$ is only locally negative definite.

- Why does the linearization method work?

  - ⋆ consider 1st order system: $\dot{x} = f(x)$
    linearize about $x = 0$: $= -ax + R(x)$

  - ⋆ assume $a > 0$ and try Lyapunov function V:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^2 \\ \dot{V}(x) &= x\dot{x} = -ax^2 + xR(x) = -x^2(a - R(x)/x) \\ &\le -x^2(a - |R(x)/x|) \end{aligned}$$

  - ⋆ but we can choose ϵ so that $|R(x)/x| < \epsilon$ whenever $|x| \le r$, so

$$\begin{aligned} \dot{V} &\le -x^2(a - \epsilon) \\ &\le -\gamma x^2 \qquad \text{with } a - \epsilon = \gamma > 0 \text{ if } |x| \le r \end{aligned}$$

$\Longrightarrow$ $\dot{V}$ negative definite for $|x|$ small enough
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Generalization to nth order systems is straightforward

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Generalization to nth order systems is straightforward

Global asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\left.\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}\right\} \text{for all } x$$
$$V(x) \to \infty \text{ as } \|x\| \to \infty$$

then $x = 0$ is globally asymptotically stable

- If $V(x) \to \infty$ as $\|x\| \to \infty$, then $V(x)$ is radially unbounded
- Test whether $V(x)$ is radially unbounded by checking if $V(x) \to \infty$ as each individual element of x tends to infinity (necessary).

Global asymptotic stability theorem

- Global asymptotic stability requires:

$$\|x(t)\| \text{ finite} \begin{cases} \text{for all } t > 0 \\ \text{for all } x(0) \end{cases}$$

$$\uparrow$$

not guaranteed by $\dot{V}$ negative definite

in addition to asymptotic stability of $x = 0$

- Hence add extra condition: $V(x) \to \infty$ as $\|x\| \to \infty$

$\updownarrow$ equiv. to

level sets $\{x \;:\; V(x) = \overline{V}\}$ are bounded

$\updownarrow$ equiv. to

$\|x\|$ is finite whenever $V(x)$ is finite

$\uparrow$

prevents $x(t)$ drifting away from 0 despite $\dot{V} < 0$

Asymptotic stability example

System: $$\dot{x}_1 = (x_2 - 1)x_1^3$$
$$\dot{x}_2 = -\frac{x_1^4}{(1 + x_1^2)^2} - \frac{x_2}{1 + x_2^2}$$

- Trial Lyapunov function $V(x) = x_1^2 + x_2^2$:

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$
$$= \underbrace{-2x_1^4 + 2x_2x_1^4 - 2\frac{x_2x_1^4}{(1 + x_1^2)^2}}_{} - 2\frac{x_2^2}{1 + x_2^2} \quad \not\leq 0$$

$\uparrow$

change V to make these terms cancel

Asymptotic stability example

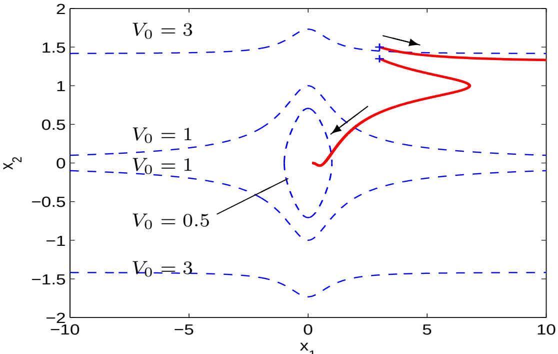
- New trial Lyapunov function $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$:

$$\dot{V}(x) = 2\left[\frac{x_1}{1+x_1^2} - \frac{x_1^3}{(1+x_1^2)^2}\right]\dot{x}_1 + 2x_2\dot{x}_2$$
$$= -2\frac{x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2} \leq 0$$

$V(x)$ positive definite, $\dot{V}(x)$ negative definite $\implies$ $x = 0$ is a.s.

But $V(x)$ not radially unbounded, so we can't conclude global asymptotic stability

State trajectories:



Summary

- Positive definite functions
- Derivative of $V(x)$ along trajectories of $\dot{x} = f(x)$
- Lyapunov's direct method for: stability
  asymptotic stability
  global stability
- Lyapunov's linearization method

Lecture 3

Convergence and invariant sets

Convergence and invariant sets

- Review of Lyapunov's direct method
- Convergence analysis using Barbalat's Lemma
- Invariant sets
- Global and local invariant set theorems

Review of Lyapunov's direct method

Positive definite functions

- If
$$V(0) = 0$$
$$V(x) > 0 \quad \text{for all } x \neq 0$$
then $V(x)$ is positive definite

- If $\mathcal{S}$ is a set containing $x = 0$ and
$$V(0) = 0$$
$$V(x) > 0 \quad \text{for all } x \neq 0,\ x \in \mathcal{S}$$
then $V(x)$ is locally positive definite (within S)

- e.g.
$$V(x) = x^\top x \qquad \leftarrow \qquad \text{positive definite}$$
$$V(x) = x^\top x(1 - x^\top x) \qquad \leftarrow \qquad \text{locally positive definite}$$
within $\mathcal{S} = \{x \ : \ x^\top x \leq \alpha\}$, $\alpha < 1$

Review of Lyapunov's direct method

System: $\dot{x} = f(x), \quad f(0) = 0$

Storage function: $V(x)$

Time-derivative of V: $\dot{V}(x) = \dfrac{\partial V}{\partial x}\dfrac{dx}{dt} = \nabla V(x)^\top \dot{x} = \nabla V(x)^\top f(x)$

– If

$$\left.\begin{array}{r} \text{(i). } V(x) \text{ is positive definite} \\ \text{(ii). } \dot{V}(x) \leq 0 \end{array}\right\} \text{ for all } x \in \mathcal{S}$$

then the equilibrium $x = 0$ is stable

– If

$$\text{(iii). } \dot{V}(x) \text{ is negative definite} \qquad \text{for all } x \in \mathcal{S}$$

then the equilibrium $x = 0$ is asymptotically stable

– If

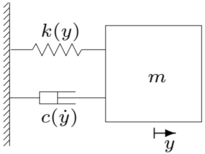
$$\begin{array}{r} \text{(iv). } S = \text{entire state space} \\ \text{(v). } V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}$$

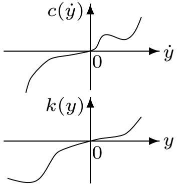
then the equilibrium $x = 0$ is globally asymptotically stable

Convergence analysis

- What can be said about convergence of $x(t)$ to 0
  if $\dot{V}(x) \leq 0$ but $\dot{V}(x)$ is not negative definite?

- Revisit m-s-d example:





Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$

Storage function:
$$V = \text{K.E.} + \text{P.E.} = \tfrac{1}{2}m\dot{y}^2 + \int_0^y k(y)\,dy$$
$$\dot{V} = -c(\dot{y})\dot{y}$$

Convergence analysis

- V is p.d. and $\dot{V} \leq 0$ so: $(y, \dot{y}) = (0, 0)$ is stable
  and $V(y, \dot{y})$ tends to a finite limit as $t \to \infty$

- but does $(y, \dot{y})$ converge to $(0, 0)$?

$\updownarrow$ equivalent to

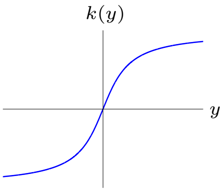
can $V(y, \dot{y})$ "get stuck" at $V = V_0 \neq 0$ as $t \to \infty$?

$\downarrow$

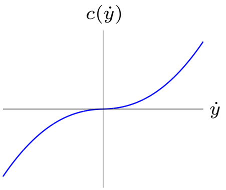
need to consider motion at points $(y, \dot{y})$ for which $\dot{V} = 0$

Example

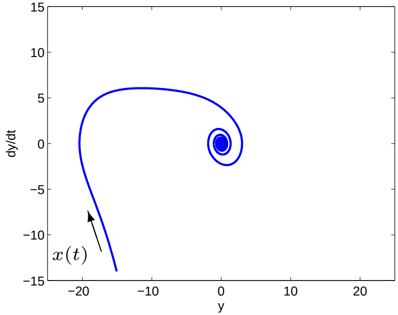
Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



$k(y) = 5\tan^{-1}(y/5)$



$c(\dot{y}) = 0.1\dot{y}(0.5 + |\dot{y}|)(2 - e^{-0.1|\dot{y}|})$



Storage function:

$$V = \frac{1}{2}\dot{y}^2 + \int_0^y 5\tan^{-1}(y/5)\,dy$$

$$\dot{V} = -c(\dot{y})\dot{y} \le 0$$

$\Downarrow$

$\dot{V} = 0$ when $\dot{y} = 0$

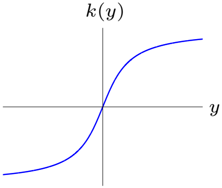
but if $k(y) \neq 0$, then $\ddot{y} \neq 0$, so $\ddot{V} \neq 0$

$\Downarrow$

V continues to decrease until $y = \dot{y} = 0$

Example

Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



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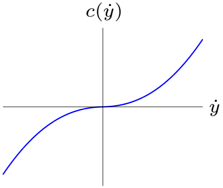

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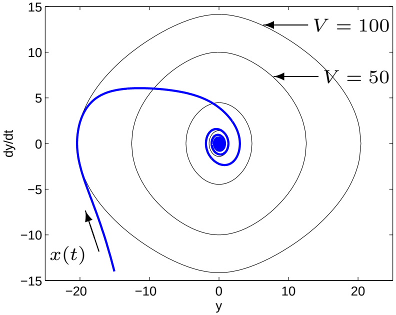


Storage function:

$$V = \tfrac{1}{2}\dot{y}^2 + \int_0^y 5\tan^{-1}(y/5)\,dy$$

$$\dot{V} = -c(\dot{y})\dot{y} \le 0$$

$$\downarrow$$

$\dot{V} = 0$ when $\dot{y} = 0$

but if $k(y) \ne 0$, then $\ddot{y} \ne 0$, so $\ddot{V} \ne 0$

$$\Downarrow$$

V continues to decrease until $y = \dot{y} = 0$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times

then $x(t)$ has to converge to $\mathcal{M}$ as $t \to \infty$

This approach is the basis of the invariant set theorems

Barbalat's Lemma

For any function $\phi(t)$, if

(i). $\displaystyle\int_0^t \phi(\tau)\,d\tau$ converges to a finite limit as $t \to \infty$

(ii). $\dot{\phi}(t)$ exists and remains finite for all t

then $\lim_{t\to\infty} \phi(t) = 0$

- ⋆ If ϕ is uniformly continuous, then
$$\int_0^t \phi(\tau)\,d\tau \to \text{constant} \quad \Longrightarrow \quad \phi(t) \to 0 \text{ as } t \to \infty$$

- ⋆ Condition (ii) ensures that $\phi(t)$ is uniformly continuous

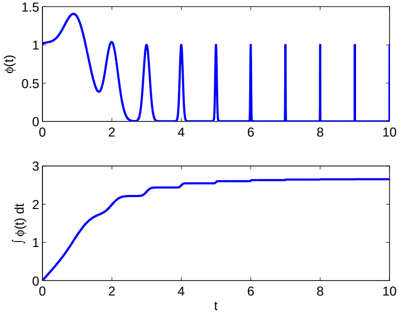
- ⋆ Without (ii) we could have
$$\left.\begin{aligned} &\int_0^t \phi(\tau)\,d\tau \to \text{constant} \\ \text{and} \quad &\phi(t) \not\to 0 \end{aligned}\right\} \text{as } t \to \infty$$

Barbalat's Lemma

Example: pulse train $\phi(t) = \sum_{k=0}^{\infty} e^{-4^k (t-k)^2}$:

$\phi(t)$:

$\int_0^t \phi(\tau)\, d\tau$:



From the plots it is clear that

$$\int_0^t \phi(s)\, ds \text{ tends to a finite limit}$$

but $\quad \phi(t) \not\to 0$ as $t \to \infty \quad$ because $\dot{\phi}(t) \to \infty$ as $t \to \infty$

Barbalat's Lemma

Apply Barbalat's Lemma to $\dot{V}\big(x(t)\big) = \phi(t) \leq 0$:

(a) Integrate:

$$\int_0^t \phi(s)\, ds = V\big(x(t)\big) - V\big(x(0)\big) \qquad \leftarrow \text{finite limit as } t \to \infty$$

(b) Differentiate:

$$\dot{\phi}(t) = \ddot{V}\big(x(t)\big) = f(x)^\top \frac{\partial^2 V}{\partial x^2}(x) f(x) + \nabla V(x)^\top \frac{\partial f}{\partial x}(x) f(x)$$

$$= \text{finite for all } t \text{ if } f(x) \text{ continuous and } V(x) \text{ continuously differentiable}$$

$$\Downarrow$$

$$\boxed{\dot{V}(x) \to 0 \text{ as } t \to \infty}$$

(a) and (b) rely on $\|x(t)\|$ remaining finite for all t,
which is implied by:

$V(x)$ positive definite
$\dot{V}(x) \leq 0$
$V(x) \to \infty$ as $\|x\| \to \infty$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
   $\to$ true whenever $\dot{V} \leq 0$ & V, f are smooth & $\|x(t)\|$ is bounded
   [by Barbalat's Lemma]

2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
   $\to$ algebra!

3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times
   $\to$ $\mathcal{M}$ must be invariant

then $x(t)$ has to converge to $\mathcal{M}$ as $t \to \infty$

This approach is the basis of the invariant set theorems

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
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   [by Barbalat's Lemma]

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This approach is the basis of the invariant set theorems

Invariant sets

- A set of points $\mathcal{M}$ in state space is invariant if

$$x(t_0) \in \mathcal{M} \quad \Longrightarrow \quad x(t) \in \mathcal{M} \quad \text{for all } t > t_0$$

Examples:

- ⋆ Equilibrium points
- ⋆ Limit cycles
- ⋆ If $\dot{V}(x) \leq 0$, then sublevel sets of $V(x)$ are invariant

↑

$\{x : V(x) \leq \alpha\}$ for constant α

- If $\dot{V}(x) \to 0$ as $t \to \infty$, then

$x(t)$ converges to an invariant set $\mathcal{M}$ contained within the set of points on which $\dot{V}(x) = 0$ as $t \to \infty$

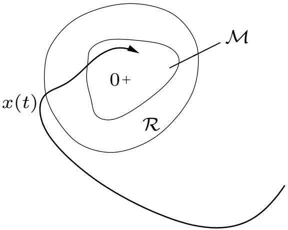
Global invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

$V(x)$ is positive definite
$\dot{V}(x) \le 0$
$V(x) \to \infty$ as $\|x\| \to \infty$

then: (i). $\dot{V}(x) \to 0$ as $t \to \infty$
(ii). $x(t) \to \mathcal{M} =$ the largest invariant set contained in $\mathcal{R}$

where $\mathcal{R} = \{x : \dot{V}(x) = 0\}$



- $\dot{V}(x)$ negative definite $\implies \mathcal{M} = 0$ (c.f. Lyapunov's direct method)
- Determine $\mathcal{M}$ by considering system dynamics within $\mathcal{R}$

Global invariant set theorem

Revisit m-s-d example

- $V(x)$ is positive definite, $V(x) \to \infty$ as $\|x\| \to \infty$, and
$$\dot{V}(y, \dot{y}) = -c(\dot{y})\dot{y} \leq 0$$

- therefore $\dot{V} \to 0$, implying $\dot{y} \to 0$ as $t \to \infty$
i.e. $\mathcal{R} = \{(y, \dot{y}) : \dot{y} = 0\}$

- but $\dot{y} = 0$ implies $\ddot{y} = -k(y)/m$

- therefore $\ddot{y} \neq 0$ unless $y = 0$, so $\dot{y}(t) = 0$ for all t only if $y(t) = 0$
i.e. $\mathcal{M} = \{(y, \dot{y}) : (y, \dot{y}) = (0, 0)\}$

$$\Downarrow$$

$(y, \dot{y}) = (0, 0)$ is a globally asymptotically stable equilibrium!

Local invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

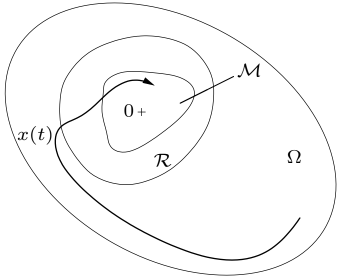
> the sublevel set $\Omega = \{x : V(x) \le \alpha\}$ is bounded for some α
> and $\dot{V}(x) \le 0$ whenever $x \in \Omega$

then: (i). Ω is an invariant set

(ii). $x(0) \in \Omega \quad \Longrightarrow \quad \dot{V}(x) \to 0$ as $t \to \infty$

(iii). $x(t) \to \mathcal{M} =$ largest invariant set contained in $\mathcal{R} \cap \Omega$

where $\mathcal{R} = \{x : \dot{V}(x) = 0\}$



Local invariant set theorem

- $V(x)$ doesn't have to be positive definite or radially unbounded

- Result is based on Barbalat's Lemma applied to $\dot{V}$

  $\uparrow$

  applies here because boundedness of Ω implies $\|x(t)\|$ finite for all t
  since $x(0) \in \Omega$ and $\dot{V} \leq 0$

- Ω is a region of attraction for $\mathcal{M}$

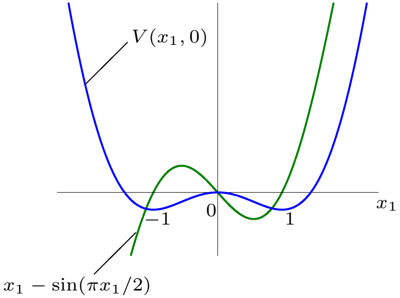
Example: local invariant set theorem

- Second order system: $$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(x_1-1)^2x_2^3 - x_1 + \sin(\pi x_1/2)\end{aligned}$$
- Equilibrium points: $(x_1, x_2) = (0,0), (1,0), (-1,0)$
- Trial storage function:
$$V(x) = \tfrac{1}{2}x_2^2 + \int_0^{x_1} \big(y - \sin(\pi y/2)\big)\, dy$$

V is not positive definite
but $V(x) \to \infty$ if $x_1 \to \infty$ or $x_2 \to \infty$

$\Downarrow$

sublevel sets of V are bounded



Example: local invariant set theorem

- Differentiate: $\dot{V}(x) = -(x_1 - 1)^2 x_2^4 \leq 0$

$$\dot{V}(x) = 0 \quad \iff \quad x \in \mathcal{R} = \{x : x_1 = 1 \;\text{ or }\; x_2 = 0\}$$

- From the system model, $x \in \mathcal{R}$ implies:

$$\begin{aligned} x_1 = 1 \quad &\implies \quad (\dot{x}_1, \dot{x}_2) = (x_2, 0) \\ &\text{and} \\ x_2 = 0 \quad &\implies \quad (\dot{x}_1, \dot{x}_2) = (0, \sin(\pi x_1/2) - x_1) \end{aligned}$$

therefore $\begin{cases} x(t) \text{ remains on line } x_1 = 1 \text{ only if } x_2 = 0 \\ x(t) \text{ remains on line } x_2 = 0 \text{ only if } x_1 = 0,\ 1 \text{ or } -1 \end{cases}$

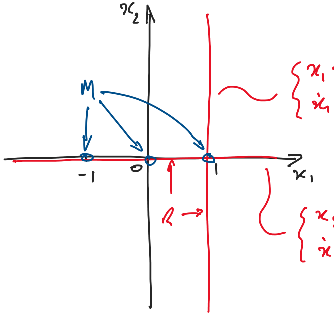
$$\implies \mathcal{M} = \{(0,0), (1,0), (-1,0)\}$$

Example: local invariant set theorem

SYSTEM:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -(x_1-1)^2 x_2^3 - x_1 + \sin(\pi x_1/2)$$



$\begin{cases} x_1 = 1 \\ \dot{x}_1 = x_2 = 0 \text{ iff } x_2 = 0 \end{cases}$

$\begin{cases} x_2 = 0 \\ \dot{x}_2 = -x_1 + \sin(\pi x_1/2) = 0 \text{ iff } x_1 = 0, \pm 1 \end{cases}$

Example: local invariant set theorem

- Apply the local invariant set theorem to any sublevel set $\Omega = \{x : V(x) \leq \alpha\}$ containing $x(0)$:

$$\left.\begin{array}{l} \Omega \text{ is bounded} \\ \dot{V} \leq 0 \end{array}\right\} \implies x(t) \to \mathcal{M} = \{(0,0), (1,0), (-1,0)\} \text{ as } t \to \infty$$

- For any given $x(0)$, we can choose sufficiently large α so that $\Omega = \{x : V(x) \leq \alpha\}$ contains $x(0)$

  so $x(t) \to \mathcal{M} = \{(0,0), (1,0), (-1,0)\}$ as $t \to \infty$ for all $x(0)$

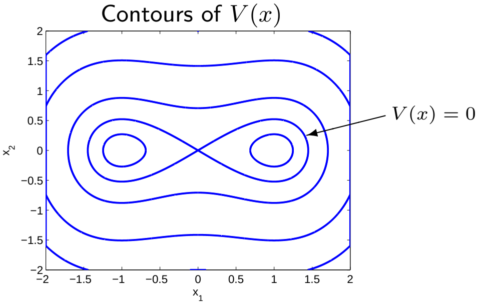
Can we find more precise limits for $x(t)$?

Example: local invariant set theorem

We have shown $x(t)$ converges asymptotically to $(0,0)$, $(1,0)$ or $(-1,0)$ but:

(a). $x = (0,0)$ is unstable since the linearization at $(0,0)$ has poles $\pm\sqrt{\frac{\pi}{2} - 1}$

(b). $V(x)$ has sublevel sets that contain only $(1,0)$ or $(-1,0)$



apply the local invariant set theorem to $\Omega = \{x : V(x) \le \alpha\}$ for $\alpha < 0$

$\downarrow$

$x = (1,0)$, $x = (-1,0)$ are stable equilibrium points

Summary

- Convergence analysis using Barbalat's lemma
- Invariant sets
- Invariant set methods for convergence analysis:

  local invariant set theorem

  global invariant set theorem

Lecture 4

Linear systems, passivity, and the circle criterion

Linear systems, passivity, and the circle criterion

- Summary of stability methods
- Lyapunov functions for linear systems
- Passive linear systems
- The circle criterion

Summary of stability methods

- Linearization method

$$\dot{x} = Ax \text{ is strictly stable, } A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$
$$\Downarrow$$
$$x = 0 \text{ locally asymptotically stable}$$

- Lyapunov's direct method

$V(x)$ locally p.d.
$\dot{V}(x) \leq 0$ locally
$\Downarrow$
$x = 0$ stable

$V(x)$ locally p.d.
$\dot{V}(x)$ locally n.d.
$\Downarrow$
$x = 0$ locally asymptotically stable

$V(x)$ p.d.
$\dot{V}(x)$ n.d.
$V(x) \to \infty$ as $\|x\| \to \infty$
$\Downarrow$
$x = 0$ globally asymptotically stable

- Invariant set theorems

$V(x)$ p.d.
$\dot{V}(x) \leq 0$
$V(x) \to \infty$ as $\|x\| \to \infty$

$\Omega = \{x \ : \ V(x) \leq V_0\}$ bounded
$\dot{V}(x) \leq 0$ for all $x \in \Omega$

$$\Downarrow$$
$$x(t) \text{ converges to the union of invariant sets contained in } \{x \ : \ \dot{V}(x) = 0\}$$

Summary of stability methods

- Instability theorems analogous to Lyapunov's direct method, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \text{ p.d.} \end{array}\right\} \quad \Longrightarrow \quad x = 0 \text{ unstable}$$

- Lyapunov stability criteria are only sufficient, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \not\leq 0 \end{array}\right\} \quad \not\Longrightarrow \quad x = 0 \text{ unstable}$$

  since some other $V(x)$ demonstrating stability may exist

- Converse theorems

$$x = 0 \text{ stable} \quad \Longrightarrow \quad V(x) \text{ demonstrating stability exists}$$

  since we can swap premises and conclusions in Lyapunov's direct method

. . . but there is no general method for constructing $V(x)$

Linear systems

- For linear systems, consider quadratic storage functions $V(x) = x^\top P x$

  If $\dot{x} = Ax$ is strictly stable then $\exists P$ such that: $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite

- Only need consider symmetric P

$$x^\top P x = \tfrac{1}{2} x^\top P x + \tfrac{1}{2} x^\top P^\top x = \tfrac{1}{2} x^\top \underbrace{(P + P^\top)}_{\text{SYMMETRIC}} x$$

- Need $\lambda(P) > 0$ for positive definite $V(x) = x^\top P x$

  $P = U \Lambda U^\top$ — eigenvector/value decomposition

  $\Downarrow$

  $x^\top P x = z^\top \Lambda z$ — $z = U^\top x$

  $\Downarrow$

  $x^\top P x$ positive definite iff $\Lambda_{ii} > 0$ — notation: $P \succ 0$ or "P is a positive definite matrix"

Linear systems

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  If $\dot{x} = Ax$ is strictly stable then $\exists P$ such that: $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite

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- Need $\lambda(P) > 0$ for positive definite $V(x) = x^\top P x$

$$P = U \Lambda U^\top \qquad \text{eigenvector/value decomposition}$$

$$\Downarrow$$

$$x^\top P x = z^\top \Lambda z \qquad z = U^\top x$$

$$\Downarrow$$

$x^\top P x$ positive definite iff $\Lambda_{ii} > 0$

$\left\{ \begin{array}{l} \text{notation: } P \succ 0 \\ \text{or "}P\text{ is a positive definite matrix"} \end{array} \right.$

Linear systems

- A systematic method for computing P

$$\left.\begin{aligned} \dot{x} &= Ax \\ V(x) &= x^\top P x \end{aligned}\right\} \implies \begin{aligned} \dot{V}(x) &= x^\top P \dot{x} + \dot{x}^\top P x \\ &= x^\top (PA + A^\top P) x \end{aligned}$$

$\therefore x = 0$ is globally asymptotically stable if, for some Q:

$$\boxed{PA + A^\top P = -Q \qquad Q = Q^\top \succ 0}$$

Lyapunov matrix equation

- Pick $Q \succ 0$ and solve $PA + A^\top P = -Q$ for P, then

$$\boxed{\mathrm{Re}[\lambda(A)] < 0 \iff \begin{array}{l}\text{unique solution for } P \\ \text{and } P = P^\top \succ 0\end{array}}$$

Linear systems

CLAIM : $PA + A^T P = -Q$ HAS A UNIQUE SOLUTION $P > 0$ FOR EVERY $Q > 0$
IF AND ONLY IF $Re[\lambda(A)] < 0$

CLAIM: $PA + A^T P = -Q$ HAS A UNIQUE SOLUTION $P > 0$ FOR EVERY $Q > 0$

IF AND ONLY IF $Re[\lambda(A)] < 0$

PROOF: LET $\dot{x} = Ax$ AND $V = \frac{1}{2} x^T P x$

① IF $PA + A^T P = -Q$ WITH $P, Q > 0$ THEN:

V IS POSITIVE DEFINITE

AND $\dot{V} = \frac{1}{2} x^T (A^T P + PA) x = -\frac{1}{2} x^T Q x$ IS NEGATIVE DEFINITE

SO $Re[\lambda(A)] < 0$ BY LYAPUNOV'S DIRECT METHOD

② IF $Re[\lambda(A)] < 0$ THEN $x(t) = e^{At} x(0)$ AND $\dot{V} = -\frac{1}{2} x^T Q x$ IMPLIES

$$\int_0^\infty \dot{V}(t)\, dt = -\frac{1}{2} x^T(0) \int_0^\infty e^{A^T t} Q e^{At}\, dt\, x(0)$$

$$\therefore\ V(0) - \underbrace{\lim_{t \to \infty} V(t)}_{=0} = x^T(0) \cdot \underbrace{\frac{1}{2} \int_0^\infty e^{A^T t} Q e^{At}\, dt}_{=P} \cdot x(0)$$

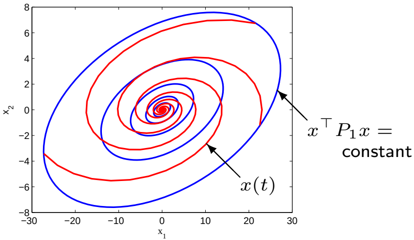
SO $V = x^T P x$ AND $P > 0$ IF $Q > 0$

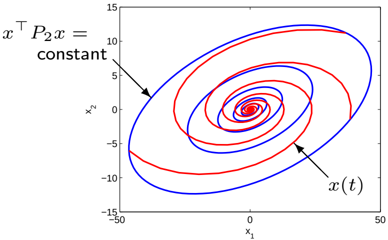
Example: Lyapunov matrix equation

Stable linear system $\dot{x} = Ax$: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -16 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\qquad \lambda(A) = -1 \pm i\sqrt{15}$

Choose Q and solve $PA + A^\top P = -Q$ for P:

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\Rightarrow\; P_1 = \begin{bmatrix} 0.33 & -0.5 \\ -0.5 & 4.25 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.41 & -0.19 \\ -0.19 & 0.11 \end{bmatrix} \;\Rightarrow\; P_2 = \begin{bmatrix} 0.12 & -0.21 \\ -0.21 & 1.67 \end{bmatrix}$$





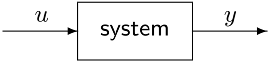
any choice of $Q \succ 0$ gives $P \succ 0$ if A is strictly stable

(but not every $P \succ 0$ gives $Q \succ 0$)

Passive systems

Systematic method for constructing storage functions

based on the input-output representation of a system:



input: u
output: y

The system mapping u to y is:

– Passive if

$$\dot{V} = yu - g \quad \text{with} \quad V(t) \geq 0, \quad g(t) \geq 0$$

here V is the "storage function"

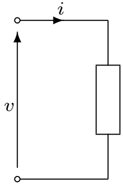
– Strictly passive* if it is passive with

$$\int_0^t g\,dt \geq \epsilon \int_0^t u^2\,dt \qquad \text{for all } u\text{, for all } t > 0\text{, and some } \epsilon > 0$$

(*some other names for this property: "strictly input passive" or "dissipative with dissipation ϵ")

Passive systems

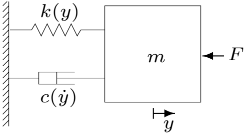
- Passivity is motivated by electrical networks with no internal power generation



input: i
output: v } stored energy:

$$V = \int_0^t vi \, dt \geq 0$$
$$\dot{V} = iv = \text{net power input}$$

- Passive mechanical systems (robotics, automotive, aerospace . . .)
  e.g. passive m-s-d system mapping input F to output $\dot{y}$:



$$m\ddot{x} + c(\dot{x}) + k(x) = F$$
$$y\,k(y) \geq 0$$
$$\dot{y}\,c(\dot{y}) \geq 0$$

$$V = \frac{1}{2}m\dot{y}^2 + \int_0^y k(x)\,dx \quad \Longrightarrow \quad \dot{V} = F\dot{y} - \dot{y}c(\dot{y})$$

Passive systems

Passivity is closely related to Lyapunov stability:

- Storage function for a passive system:

$$\dot{V} \leq yu$$

  rate of energy increase is less than input power

- Lyapunov function $V(x)$ for a stable system:

$$\dot{V} \leq 0$$

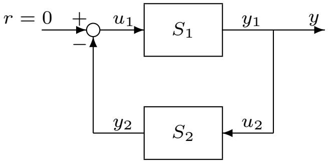
  energy decreases with time

- Note that passivity doesn't require $V(x)$ to be positive definite in general

Passive systems

Passivity allows storage functions to be determined for feedback systems

(1) Closed-loop system with passive subsystems S_1 and S_2:



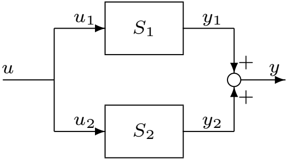
$$S_1: \quad V_1 \geq 0 \quad \dot{V}_1 = y_1 u_1 - g_1$$
$$S_2: \quad V_2 \geq 0 \quad \dot{V}_2 = y_2 u_2 - g_2$$

$$V_1 + V_2 \geq 0$$
$$\begin{aligned} \dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &= y_1(-y_2) + y_2 y_1 - g_1 - g_2 \\ &= -g_1 - g_2 \\ &\leq 0 \end{aligned}$$

$\implies$ $V = V_1 + V_2$ is a Lyapunov function for the closed-loop system if V is a positive definite function of the state of (S_1, S_2)

Interconnected passive systems

(2) Parallel connection:

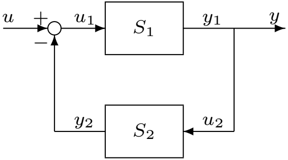


$$
\begin{aligned}
V_1 + V_2 &\geq 0 \\
\dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\
&= (y_1 + y_2) u - g_1 - g_2 \\
&= y u - g_1 - g_2
\end{aligned}
$$

$$\Downarrow$$

Overall system from u to y is passive

(3) Feedback connection:



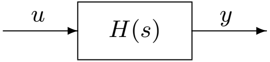
$$
\begin{aligned}
V_1 + V_2 &\geq 0 \\
\dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\
&= y(u - y_2) + y_2 y - g_1 - g_2 \\
&= y u - g_1 - g_2
\end{aligned}
$$

$$\Downarrow$$

Overall system from u to y is passive

Passive linear systems

Transfer function : $\dfrac{Y(s)}{U(s)} = H(s)$



- H is passive if and only if

(i).	$\mathrm{Re}(p_i) \leq 0$	for all poles p_i of $H(s)$
(ii).	$\mathrm{Re}\big[H(j\omega)\big] \geq 0$	for all $0 \leq \omega \leq \infty$

⋆ H must be stable, otherwise $V(t) = \displaystyle\int_0^t yu\,dt$ is not defined for all u

⋆ From Parseval's theorem:

$$\mathrm{Re}\big[H(j\omega)\big] \geq 0 \quad \iff \quad \int_0^t yu\,dt \geq 0 \text{ for all } u(t) \text{ and } t$$

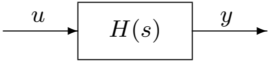
↑

frequency domain condition for passivity

H is called a "positive real" system

Passive linear systems

Transfer function : $\dfrac{Y(s)}{U(s)} = H(s)$



- H is strictly passsive (also called "strictly positive real") if $\mathrm{Re}(p_i) < 0$ and

$$\mathrm{Re}\big[H(j\omega)\big] > 0 \qquad \text{for all } 0 \le \omega < \infty$$

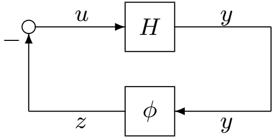
- Kalman-Yakubovich-Popov (KYP) Lemma implies

> If H is strictly passive, then there exist $P \succ 0$ and $Q \succ 0$ such that
>
> $$V = x^\top P x \text{ and } \dot{V} = yu - x^\top Q x$$

⋆ x is the state of any controllable & observable state space realization of H

⋆ $x = 0$ is globally asymptotically stable with any passive output feedback law $u = -\phi(y)$

Linear system + static nonlinearity



H linear: $\quad \dfrac{Y(s)}{U(s)} = H(s)$

ϕ static nonlinearity: $\quad z = \phi(y)$

What are the conditions on H and ϕ for closed-loop stability?

- A common problem in practice, due to e.g.
  - ⋆ actuator saturation (valves, dc motors, etc.)
  - ⋆ sensor nonlinearity

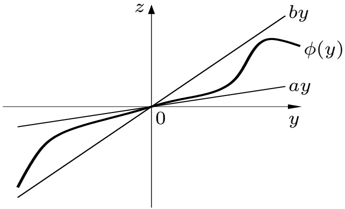
- Determine closed-loop stability given:

ϕ belongs to sector $[a, b]$

$$\Updownarrow$$

$$a \leq \frac{\phi(y)}{y} \leq b$$

"Absolute stability problem"



Linear system + static nonlinearity

- Aizerman's conjecture (1949):

  Closed-loop system is stable if stable for $\phi(y) = ky$, *for all constant* $k \in [a, b]$

  this is false (it's necessary but not sufficient)

- Sufficient conditions for closed-loop stability:

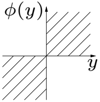
  Popov criterion (1960) / Circle criterion } based on passivity

- The passivity approach:

  (1). If H is strictly passive, then $P, Q \succ 0$ exist so that $V = x^\top P x$ satisfies

  $$\begin{aligned}\dot V &= yu - x^\top Q x \\ &= -y\phi(y) - x^\top Q x\end{aligned}$$

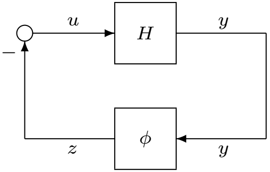
  (2). If ϕ belongs to sector $[0, \infty)$, then: $y\phi(y) \geq 0$



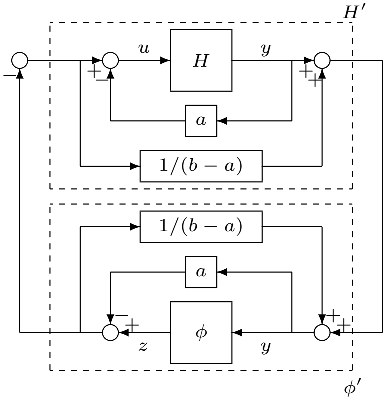
  $$\begin{aligned}\text{(1) \& (2)} \quad &\implies \quad \dot V \leq -x^\top Q x \\ &\implies \quad x = 0 \text{ is globally asymptotically stable}\end{aligned}$$

Circle criterion

Use loop transformations to generalize the approach for $\begin{cases} H \text{ not passive} \\ \phi \notin [0, \infty) \end{cases}$



$\longleftrightarrow$ equiv. to



$\phi \in [a, b] \quad a, b$ arbitrary

$$\phi \in [a, b] \implies \phi' \in [0, \infty]$$

$$H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b - a}$$

Circle criterion

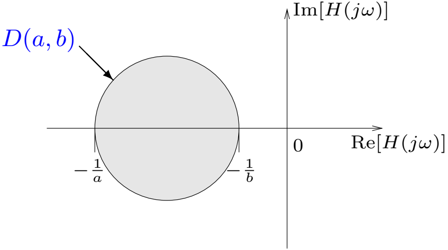
To make $H'(j\omega) = \dfrac{H(j\omega)}{1 + aH(j\omega)} + \dfrac{1}{b-a}$ strictly passive, need:

(i). H' stable $\iff$ $\dfrac{H(j\omega)}{1 + aH(j\omega)}$ stable

$$\Updownarrow$$

Nyquist plot of $H(j\omega)$ goes through ν anti-clockwise encirclements of $-1/a$ as ω goes from $-\infty$ to ∞ (ν = no. poles of $H(j\omega)$ in RHP)

(ii). $\mathrm{Re}\big[H'(j\omega)\big] > 0 \iff \begin{cases} H(j\omega) \text{ lies outside } D(a,b) & \text{if } ab > 0 \\ H(j\omega) \text{ lies inside } D(a,b) & \text{if } ab < 0 \end{cases}$

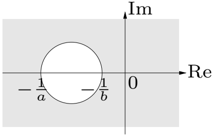


Graphical interpretation of circle criterion

$x = 0$ is globally asymptotically stable if:

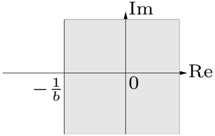
- $0 < a < b$ and

  $H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(a, b)$



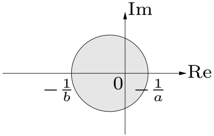
- $b > a = 0$ and

  $H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



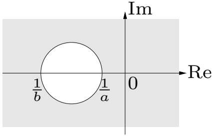
- $a < 0 < b$ and

  $H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



- $a < b < 0$ and

  $-H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(-b, -a)$



Circle criterion

- Circle criterion is equivalent to Nyquist criterion for $a = b > 0$

$$\uparrow$$

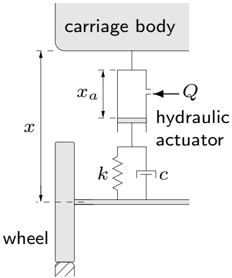
$$D(a,b) = -\frac{1}{a} \quad \text{(single point)}$$

- Circle criterion is only sufficient for closed-loop stability for general a, b

- Results apply to time-varying static nonlinearity: $\phi(y, t)$

Example: Active suspension system

- Active suspension system for high-speed train:



$$
\begin{aligned}
Q &= \phi(u) \\
\dot{x}_a &= Q/A
\end{aligned}
$$

u : valve input signal
Q : flow rate
ϕ : valve characteristics, $\phi \in [0.005, 0.1]$
A : actuator working area

- Force exerted by suspension system on carriage body: F_{susp}

$$
\begin{aligned}
F_{\text{susp}} &= k(x_a - x) + c(\dot{x}_a - \dot{x}) \\
&= \Big(k \int^t Q\, dt + cQ\Big)/A - kx - c\dot{x}, \qquad Q = \phi(u)
\end{aligned}
$$

- Design controller to compensate for the effects of (constant) unknown load on displacement x despite uncertain valve characteristics $\phi(u)$.

Active suspension system contd.

- Dynamics:

$$F_{\mathsf{susp}} - F = m\ddot{x}$$

$$\implies \quad m\ddot{x} + c\dot{x} + kx = \Big(k\int^t Q\,dt + c\,Q\Big)/A - F, \qquad Q = \phi(u)$$

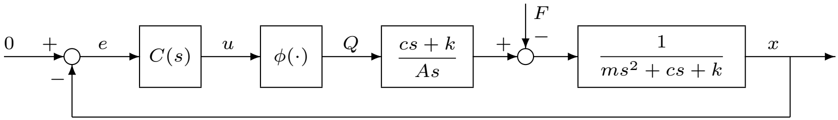
F : unknown load on suspension unit
m : effective carriage mass

- Transfer function model:

$$X(s) = \frac{cs+k}{ms^2+cs+k}\cdot\frac{Q(s)}{As} - \frac{F}{ms^2+cs+k} \qquad Q = \phi(u)$$

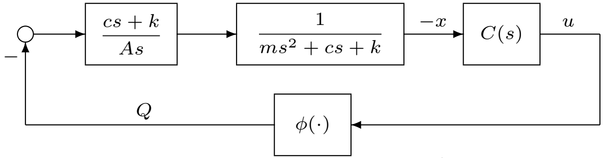
- Try linear compensator $C(s)$:

$$U(s) = C(s)E(s) \qquad e = -x, \quad \text{setpoint: } x = 0$$



Active suspension system contd.

- For constant F, we need to stabilize the closed-loop system:



linear system: $H(s) = \dfrac{cs+k}{As(ms^2+cs+k)} \cdot C(s)$

static nonlinearity: $\phi \in [0.005, 0.1]$

- P+D compensator (no integral term needed):

$$C(s) = K(1+\alpha s) \quad \Longrightarrow \quad H(s) = \frac{K(1+\alpha s)(cs+k)}{As(ms^2+cs+k)}$$

H open-loop stable ($\nu = 0$)

- From the circle criterion, closed-loop (global asymptotic) stability is ensured if:

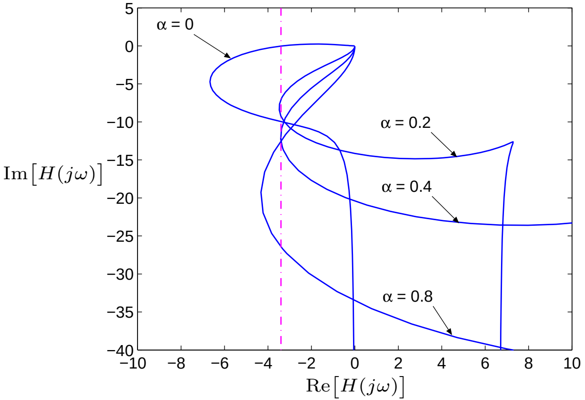
$H(j\omega)$ lies outside $D(0.005, 0.1)$

$\uparrow$

sufficient condition: $\mathrm{Re}\big[H(j\omega)\big] > -10$

Active suspension system contd.

▷ Nyquist plot of $H(j\omega)$ for $K = 1$ and $\alpha = 0, 0.2, 0.4, 0.8$:



▷ To maximize gain margin:

choose $\alpha = 0.2$ $\leftarrow$ allows for largest K

$K \leq 10/3.4 = 2.94$

Summary

At the end of the course you should be able to do the following:

- Understand the basic Lyapunov stability definitions (lecture 1)
- Analyse stability using the linearization method (lecture 2)
- Analyse stability by Lyapunov's direct method (lecture 2)
- Determine convergence using Barbalat's Lemma (lecture 3)
- Understand how invariant sets can determine regions of attraction (lecture 3)
- Construct Lyapunov functions for linear systems and passive systems (lecture 4)
- Use the circle criterion to design controllers for systems with static nonlinearities (lecture 4)