Nonlinear Systems Examples Sheet: Solutions

Mark Cannon, Hilary Term 2023

Equilibrium points

1. (a). Solving $\dot{x} = \sin^4 x - x^3 = 0$ for x gives $x = 0$ as an equilibrium point. This is the only equilibrium because there is only one point ($x = 0$) where $\sin x = x$ since

$$|\sin x| < |x| < 1 \implies |\sin x|^4 < |x|^3 \text{ for all } |x| \le 1,\ x \ne 0$$
$$|\sin x| \le 1 \implies |\sin x|^4 < |x|^3 \text{ for all } |x| > 1$$

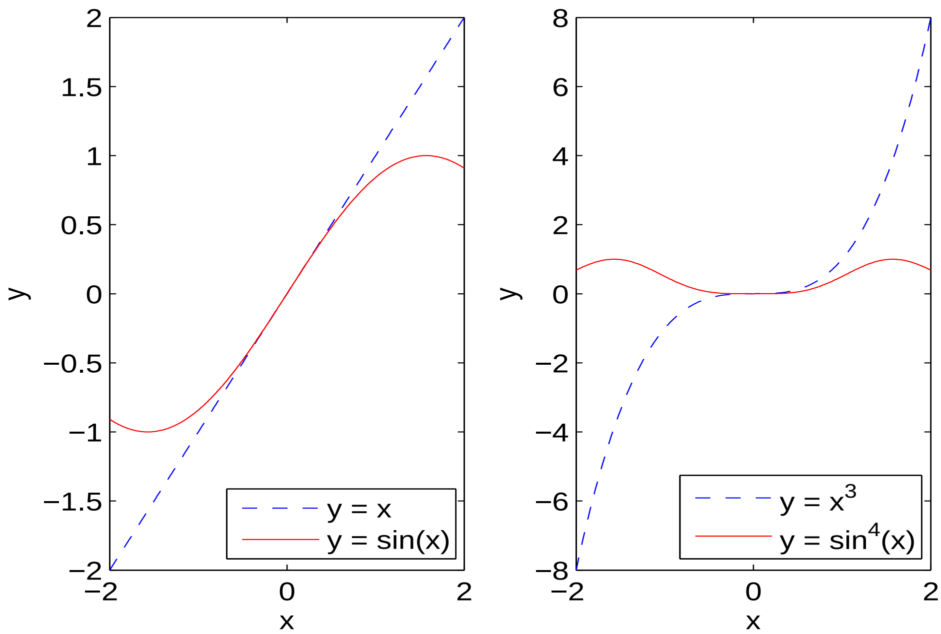


Figure 1: solution of $x^3 = \sin^4 x$ for question 1

(b). In terms of state variables $(x_1, x_2) = (x, \dot{x})$:

$$\dot{x}_1 = \dot{x} = x_2$$
$$\dot{x}_2 = \ddot{x} = -(x_1 - 1)^2 x_2^5 - x_1^2 + \sin(\pi x_1/2)$$

At an equilibrium point $\dot{x}_1 = \dot{x}_2 = 0$. But $\dot{x}_1 = 0$ implies $x_2 = 0$, so

$$\dot{x}_2 = 0 \implies x_1^2 - \sin(\pi x_1/2) = 0 \implies x_1 = 0 \text{ or } 1$$

Therefore equilibrium points are $(x_1, x_2) = (x, \dot{x}) = (0, 0)$ and $(1, 0)$.

Lyapunov's direct method, invariant sets and linearization

2. To explain the significance of constants a, b, c, we first give a derivation of the dynamics (this is not asked for in the question). The angular momentum of the craft in xyz-coordinates (Fig. 2) is given by

$$H = I\omega, \quad I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where I_x, I_y, I_z are the moments of inertia about x, y, and z-axes (assumed to be aligned with the spacecraft's principal axes). Since there is no torque acting on the craft:

$$\frac{d}{dt}(I\omega) = I\dot{\omega} + \omega \times I\omega = 0$$

(where the $\omega \times I\omega$ term is needed because xyz-coordinates are fixed to and hence rotate with the spacecraft). So the full dynamics are given by

$$\dot{\omega}_x = a\omega_y\omega_z \qquad \dot{\omega}_y = -b\omega_x\omega_z \qquad \dot{\omega}_z = c\omega_x\omega_y$$
$$a = (I_y - I_z)/I_x, \quad b = (I_x - I_z)/I_y, \quad c = (I_x - I_y)/I_z$$

and the constants a, b, c are all positive if $I_x > I_y > I_z$.

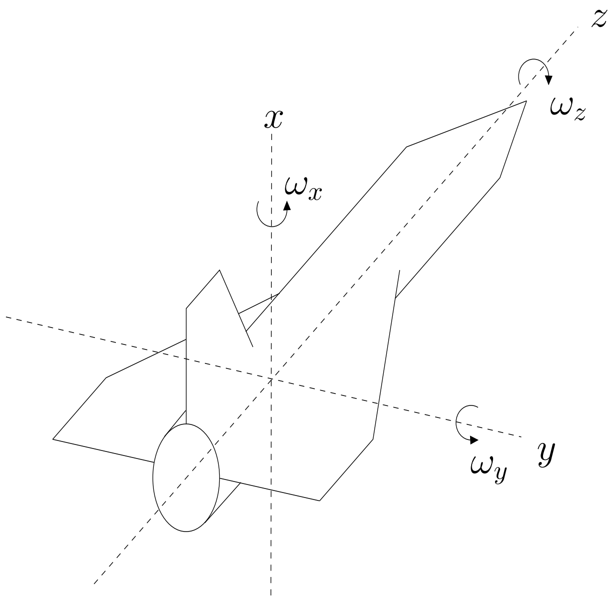


Figure 2: Rotating spacecraft.

(a). Equilibrium points: $\dot{\omega}_x = 0 \iff \omega_y = 0$ or $\omega_z = 0$, i.e. at least two of ω_x, ω_y and ω_z must be zero for $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Therefore every point in state space lying on the ω_x-axis, the ω_y-axis, or the ω_z-axis is an equilibrium point.

(b). To show stability of the equilibrium at $\omega = 0$, try $V = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$ as a Lyapunov function. Clearly V is positive definite if p, q, r are all positive. Also

$$\begin{aligned} \dot{V} &= 2(p\omega_x\dot{\omega}_x + q\omega_y\dot{\omega}_y + r\omega_z\dot{\omega}_z) \\ &= 2(pa - qb + rc)\omega_x\omega_y\omega_z \end{aligned}$$

Hence choosing p, q, r so that

$$p > 0,\ q > 0,\ r > 0,\ \text{and } pa - qb + rc = 0,$$

(which is always possible since $q = (pa + rc)/b$ is positive for any chosen positive p, r), results in $\dot{V} = 0$, implying that $\omega = 0$ is a stable equilibrium point by Lyapunov's direct method.

(c). Differentiating the function

$$V = c\omega_y^2 + b\omega_z^2 + \big[2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)\big]^2$$

(for constant ω_0) with respect to t along system trajectories yields

$$\begin{aligned} \dot{V} = {} & \underbrace{2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z}_{=0} \\ & + 2\big[2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)\big]\underbrace{(4ac\omega_y\dot{\omega}_y + 2ab\omega_z\dot{\omega}_z + 2bc\omega_x\dot{\omega}_x)}_{=0} \end{aligned}$$

i.e. $\dot{V} = 0$. Also $V = 0$ only if $\omega = (\pm\omega_0, 0, 0)$, and $V > 0$ whenever $\omega_x \neq \pm\omega_0$, $\omega_y \neq 0$ or $\omega_z \neq 0$, so that V is a locally positive definite function centered at the equilibrium $(\pm\omega_0, 0, 0)$. Therefore $\dot{V} = 0$ implies that every point on the ω_x-axis in state space is a stable equilibrium, and hence that rotation at any constant velocity about the x-axis alone is stable.

[Note that rotational motion about the z-axis is likewise stable since a, c and ω_x, ω_z can be swapped in the dynamics and in the definition of V. However rotation about the y-axis is unstable, as shown by the linearized system at $\omega = (0, \omega_0, 0)$:

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & a\omega_0 \\ 0 & 0 & 0 \\ c\omega_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

which has eigenvalues $\pm\sqrt{ac}\omega_0$ and 0, and is therefore unstable.]

3. (a). The positive definite function $V = \frac{1}{2}x^2$ has derivative:

$$\dot{V} = x\dot{x} = -xb(x)$$

which is negative definite due to $xb(x) > 0$ whenever $x \neq 0$. Therefore $x = 0$ is asymptotically stable, and since $V \to \infty$ as $x \to \infty$ it follows that $x = 0$ is globally asymptotically stable by Lyapunov's direct method.

(b). At an equilibrium point $\dot{x} = 0$. Hence $\ddot{x} = -c(x) = 0$ implies $x = 0$ since the condition $xc(x) > 0$ whenever $x \neq 0$ implies that $c(x)$ can only be equal to zero if $x = 0$. Therefore the only equilibrium point is the origin of state space: $(x, \dot{x}) = (0, 0)$.

The function $V(x, \dot{x})$ is positive definite and has derivative

$$\dot{V} = \dot{x}\ddot{x} + c(x)\dot{x} = -\dot{x}b(\dot{x}) \leq 0$$

and hence $(x, \dot{x}) = (0, 0)$ is stable by Lyapunov's direct method.

To apply the local invariant set theorem, we need to show that: (i) the level sets $\{(x, \dot{x}) : V(x, \dot{x}) \leq V_0\}$ are bounded for some V_0; (ii) $\dot{V} \leq 0$; (iii) the system dynamics are continuous and V is continuously differentiable in x and $\dot{x}$. Here (i) is satisfied because V is increasing in both x (since sign$(c(x))$ = sign(x)) and $\dot{x}$; (ii) is demonstrated above; and (iii) holds since $b(\dot{x})$, $c(x)$, $\partial V/\partial \dot{x} = \dot{x}$, and $\partial V/\partial x = c(x)$ are all continuous functions of x and $\dot{x}$. Let $\mathcal{R} = \{(x, \dot{x}) : \dot{V} = 0\}$ and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$, then

$$\mathcal{R} = \{(x, \dot{x}) : \dot{x} = 0\}$$

and since $\ddot{x} = 0$ is necessary in order that the state remains in $\mathcal{R}$, we have

$$\mathcal{M} = \mathcal{R} \cap \{(x, \dot{x}) : \ \ddot{x} = 0\} = \{(x, \dot{x}) : \ c(x) = 0\} = \{(0, 0)\}.$$

From the local invariant set theorem, $(x, \dot{x})$ therefore converges asymptotically to $\mathcal{R}$ from all initial conditions within any bounded level set of V, implying that $(0, 0)$ is asymptotically stable.

To show global asymptotic stability we need V to be radially unbounded (in order to apply the global invariant set theorem) or equivalently the level sets of V must cover the entire state space as $V_0 \to \infty$. This condition requires

$$\int^x c(s)\, ds \to \infty \text{ as } x \to \infty.$$

4. (a). The equilibrium points can be found by solving $\dot{x}_1 = \dot{x}_2 = 0$ for x_1 and x_2:

$$\begin{aligned} \dot{x}_1 = 0 \quad &\implies \quad x_2 = 0 \\ \dot{x}_1 = \dot{x}_2 = 0 \quad &\implies \quad x_1(x_1^2 - 1) = 0 \quad \implies \quad x_1 = 0, 1, -1. \end{aligned}$$

Hence the equilibrium points are $(x_1, x_2) = \{(0, 0), (1, 0), (-1, 0)\}$.

(b). The system and function V have the following properties.

(i). V, $\dot{x}_1$ and $\dot{x}_2$ are continuous functions of x_1 and x_2.

(ii). The level sets: $\{(x_1, x_2) : \ V \leq V_0\}$ are finite and V is radially unbounded since $V \to \infty$ as $|x_1| \to \infty$ and/or $|x_2| \to \infty$.

(iii). Along system trajectories, V has derivative

$$\begin{aligned} \dot{V}(x_1, x_2) &= x_2\dot{x}_2 + x_1(x_1^2 - 1)\dot{x}_1 \\ &= -x_2^2(x_1 - 1)^2 - x_1x_2(x_1^2 - 1) + x_1x_2(x_1^2 - 1) \\ &= -x_2^2(x_1 - 1)^2 \\ &\leq 0. \end{aligned}$$

Using the global invariant set theorem, (i)-(iii) imply that every state trajectory tends to an invariant set on which $\dot{V} = 0$. (The same conclusion can be reached using the local invariant set theorem, since the level sets of V can be made arbitrarily large by choosing V_0 sufficiently large.)

From (iii), $\dot{V}(x_1, x_2) = 0$ is satisfied on the lines $x_2 = 0$ and $x_1 = 1$. The invariant sets within these lines are defined by $\dot{x}_2 = 0$ (on $x_2 = 0$) and $\dot{x}_1 = 0$ (on $x_1 = 1$). But

$$\left.\begin{array}{r} x_2 = 0 \\ \dot{x}_2 = 0 \end{array}\right\} \implies x_1 = 0, 1, -1, \qquad \left.\begin{array}{r} x_1 = 1 \\ \dot{x}_1 = 0 \end{array}\right\} \implies x_2 = 0$$

and every state trajectory therefore tends asymptotically to one of the three equilibrium points identified in (a).

(c). Writing the system dynamics in the form $\dot{x} = f(x)$, $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ where the Jacobian matrix of f is

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1 \\ -2x_2(x_1 - 1) - (3x_1^2 - 1) & -(x_1 - 1)^2 \end{bmatrix},$$

the linearization of the system at $x_1 = x_2 = 0$ is given by

$$\dot{x} = Ax, \qquad A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

A has eigenvalues $-1/2 \pm \sqrt{5}/2$, and it follows that the origin is an unstable equilibrium of the nonlinear system, by Lyapunov's linearization method.

(d). V has local minimum points at $(x_1, x_2) = (-1, 0)$ and $(1, 0)$ (since

$$\nabla V = \begin{bmatrix} x_1^3 - x_1 \\ x_2 \end{bmatrix} = 0 \qquad \frac{\partial^2 V}{\partial x^2} = \begin{bmatrix} 3x_1^2 - 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} > 0$$

at $(x_1, x_2) = (-1, 0)$ and $(1, 0)$). Hence $V + \frac{1}{4}$ is locally positive definite at $(x_1, x_2) = (-1, 0)$ and $(1, 0)$, and from Lyapunov's direct method these equilibrium points are therefore stable because $\dot{V} \leq 0$.

Other approaches for (d): The equilibrium at $(-1, 0)$ can be shown to be stable using the linearization method, since the linearization at this point is stable. However the linearization about $(1, 0)$ has eigenvalues $\pm i\sqrt{2}$, and therefore does not allow any conclusions to be made about the stability of this equilibrium for the nonlinear system.

5. (a). Using matrices A, B, K and the given matrix P we get (2 marks):

$$Q = -(A - BK)^T P - P(A - BK) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

where

$$\begin{aligned} \text{eig}(P) = \lambda \ &: \ \lambda^2 - 3\lambda + 1 = 0 \implies \lambda = \tfrac{3}{2} \pm \tfrac{\sqrt{5}}{2} \\ \text{eig}(Q) = \lambda \ &: \ \lambda^2 - 4\lambda + 3 = 0 \implies \lambda = 1, 3 \end{aligned}$$

The equilibrium $x = 0$ is locally asymptotically stable since:

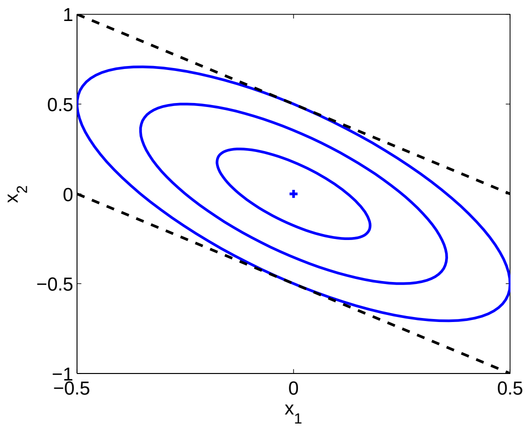
- the linearized closed loop system about $x = 0$ is $\dot{x} = (A - BK)x$
- $(A - BK)^T P + P(A - BK) = -Q$ for positive definite P, Q implies $\dot{x} = (A - BK)x$ is stable, i.e. Re[eig$(A - BK)$] < 0
- so the nonlinear closed loop system is locally a.s.

(b). From $V = x^T P x$ and $\dot{x} = (A - BK)x - x(Kx)$ we get

$$\begin{aligned} \dot{V} &= x^T \big[(A - BK)^T P + P(A - BK)\big] x - (Kx) x^T (P + P) x \\ &= -x^T Q x - 2(Kx)\, x^T P x \\ &\le -x^T Q x + 2|Kx|\, x^T P x \end{aligned}$$

But $x^T P x - x^T Q x = x^T (P - Q) x = -x_2^2 \le 0$,
so $\dot{V} \le -x^T Q x + 2|Kx|\, x^T Q x$.

(c). $\dot{V} \le -x^T Q x (1 - 2|Kx|)$, so $\dot{V}$ is negative definite in the region where $|Kx| < \frac{1}{2}$, which is the strip between the dashed lines in the figure below.



Any level set of V contained entirely within this strip is invariant and hence is a region of attraction for $x = 0$.

The level sets Ω are ellipsoidal, centred on the origin, and decrease in size as α is reduced. Hence Ω must be invariant for small enough α.

Linear and passive systems

6. Let $\Phi = A + \mu I$, then $A^T P + PA + 2\mu P = -Q$ implies

$$\Phi^T P + P\Phi = A^T P + PA + 2\mu P = -Q,$$

so $P, Q > 0$ imply that Re$\{$eig$(\Phi)\} < 0$, so that Re$\{$eig$(A + \mu I)\} < 0$, and therefore Re$\{$eig$(A)\} < -\mu$
(since $A = V\Lambda V^{-1} \implies \Phi = V(\Lambda - \mu I)V^{-1}$).

7. (a). Differentiating V_1 with respect to t gives:

$$\dot{V}_1 = \frac{x_2 e}{L(x_2)} - \frac{R_1}{L^2(x_2)} x_2^2 = \dot{x}_1 e - \frac{R_1}{L^2(x_2)} x_2^2$$

and since $V \geq 0$, this implies that the dynamic system with e as input and $\dot{x}_1$ as output is passive (in fact it is dissipative).

(b). Let x_3 and x_4 be respectively the charge on the capacitor and flux in the inductor in the right-hand branch of the circuit, and define

$$V_2(x_3, x_4) = \int_0^{x_4} \frac{x}{L(x)}\, dx \; + \int_0^{x_3} \frac{x}{C(x)}\, dx.$$

Differentiating w.r.t. t gives $\dot{V}_2 = \dot{x}_3 e - R_2 x_4^2 / L^2(x_4)$. Therefore, defining $V = V_1 + V_2$ and using the fact that $\dot{x}_1 + \dot{x}_3 = i$ (since the currents in the two branches of the circuit must sum to i), we get

$$V = \int_0^{x_2} \frac{x}{L(x)}\, dx + \int_0^{x_4} \frac{x}{L(x)}\, dx + \int_0^{x_1} \frac{x}{C(x)}\, dx + \int_0^{x_3} \frac{x}{C(x)}\, dx$$
$$\dot{V} = ie - \frac{R_1}{L^2(x_2)} x_2^2 - \frac{R_2}{L^2(x_4)} x_4^2.$$

and $V \geq 0$ since $V_1, V_2 \geq 0$.

Opening the switch forces $i = 0$, so

$$\dot{V} = -\frac{R_1}{L^2(x_2)} x_2^2 - \frac{R_2}{L^2(x_4)} x_4^2$$

and since the level sets $\{(x_1, x_2, x_3, x_4) : V \leq \bar{V}\}$ are bounded (when $\bar{V}$ is sufficiently small), it follows from the local invariant set theorem that the system is (locally) asymptotically stable.

Specifically, $x = (x_1, x_2, x_3, x_4)$ must converge to the largest invariant set within the set of states such that $\dot{V} = 0$, i.e. $x_2 = x_4 = 0$ (so the currents $\dot{x}_1$ and $\dot{x}_3$ must converge to zero) and $\dot{x}_2 = \dot{x}_4 = 0$, implying that x converges asymptotically to a steady state such that $x_1/C(x_1) = x_3/C(x_3)$ and $(x_2, x_4) = (0, 0)$. This asymptotic stability property is global if V_1 and V_2 are radially unbounded. Note that the equilibrium points to which the system can converge lie on a 2-dimensional surface in state space (defined by $x_1/C(x_1) = x_3/C(x_3)$) and also that the same analysis can be applied to any number of LCR branches connected in parallel.

8. (a). The rectangular region containing $G(j\omega)$ lies within $D(a, b)$ if $a = -\frac{1}{3}$ and $b = \frac{1}{2}$, since $D(a, b)$ is then just touching its corners (Fig. 3). The open-loop system is stable, and the circle criterion therefore implies that the closed-loop system with $u = -\phi(y)$ will be asymptotically stable if ϕ lies in the sector $[-\frac{1}{3}, \frac{1}{2}]$.

Clearly this is not the only sector bound for ϕ for which the closed-loop system is guaranteed to be stable by the circle criterion. In fact a family of discs $D(a, b)$ containing $G(j\omega)$ is generated as a is increased from $-1/3$, and to allow for the largest possible value of b we need to set $a = 0$ and $b = -1$, corresponding to sector bounds $\phi \in [0, 1]$.

(b). Closed-loop stability does not apply to nonlinearities ϕ bounded by the union of the two sectors defined in part (a), i.e. $[-\frac{1}{3}, 1]$, since this includes nonlinearities not belonging to either of the sectors $[-\frac{1}{3}, \frac{1}{2}]$ and $[0, 1]$. In particular, the disc centred on the real axis and intersecting the real axis at -1 and 3 does not entirely contain the box in which $G(j\omega)$ is known to lie, so it cannot be concluded from the circle criterion that the closed loop system will be stable.

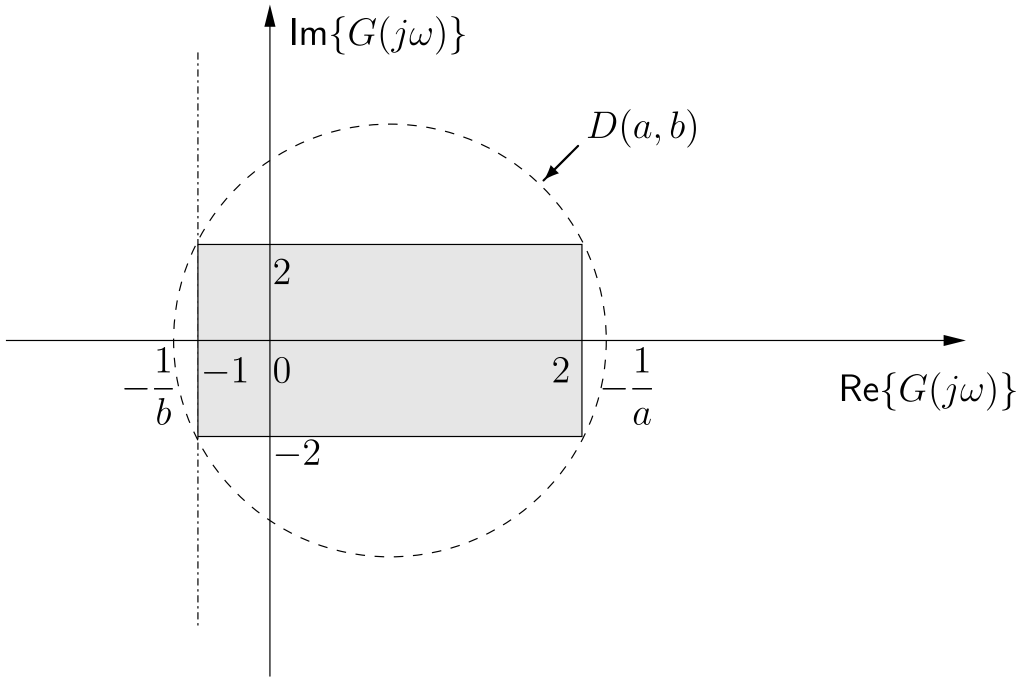


Figure 3: Bounds on the Nyquist plot of $G(j\omega)$.