Lecture 1

Introduction and Concepts of Stability
Organisation

▷ 4 lectures: week 1  
   Mon 10-11 am  
   Tue 10-11 am  
   Thu 10-11 am  
   Fri 10-11 am

▷ 1 class: week 6  
   Thu 11-12 pm
or week 7  
   Thu 10-11 am
Course outline

1. Types of stability
2. Linearization
3. Lyapunov’s direct method
4. Regions of attraction
5. Linear systems and passive systems
Books

  ★ Stability
  ★ Interconnected systems and passive systems

  ★ Stability
  ★ Passive systems

  ★ Stability & passivity (more technical detail)
Why use nonlinear control?

- Real systems are nonlinear
  - friction, non-ideal components
  - actuator saturation
  - sensor nonlinearity

- Analysis via linearization
  - accuracy of approximation?
  - conservative?

- Account for nonlinearities in high performance applications

- Account for nonlinearities if linear models inadequate
  - large operating region
  - model properties change at linearization point
Linear vs nonlinear system properties

Free response

**Linear system**

\[ \dot{x} = Ax \]

- Unique equilibrium point:
  \[ Ax = 0 \iff x = 0 \]

- Stability independent of initial conditions

**Nonlinear system**

\[ \dot{x} = f(x) \]

- Multiple equilibrium points
  \[ f(x) = 0 \]

- Stability dependent on initial conditions
Linear systems reminder

Linear system free response

\[ \dot{x} = Ax \]

Eigen-decomposition: \[ Av_i = v_i \lambda_i \]

let \[ V = \begin{bmatrix} v_1, \ldots, v_n \end{bmatrix} \]
\[ \Lambda = \begin{bmatrix} \lambda_1 & \cdots & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \lambda_n \end{bmatrix} \]

then \[ A = V \Lambda V^{-1} \] (if \( V^{-1} \) exists)

\[ \Rightarrow \quad \dot{z} = \Lambda z, \quad z = V^{-1}x \]
\[ z(t) = e^{\Lambda t} z(0) \]

\[ \Rightarrow \quad x(t) = Ve^{\Lambda t}V^{-1}x(0) = e^{At}x(0) \]

System is stable if \( \text{Re}(\lambda_i) < 0 \ \forall i \)
Linear vs nonlinear system properties

Forced response

**Linear system**

\[ \dot{x} = Ax + Bu \]

- \( \|u\| \) finite \( \Rightarrow \) \( \|x\| \) finite if open-loop stable

- Frequency response:
  \( u = U \sin \omega t \) \( \Rightarrow \) \( x = X \sin(\omega t + \phi) \)

- Superposition:
  \( u = u_1 + u_2 \) \( \Rightarrow \) \( x = x_1 + x_2 \)

**Nonlinear system**

\[ \dot{x} = f(x, u) \]

- \( \|u\| \) finite \( \not\Rightarrow \) \( \|x\| \) finite

- No frequency response
  \( u = U \sin \omega t \not\Rightarrow x \) sinusoidal

- No linear superposition
  \( u = u_1 + u_2 \not\Rightarrow x = x_1 + x_2 \)
Linear system free response

\[ \dot{x} = Ax \]

Eigen-decomposition: \( Av_i = v_i \lambda_i \)

Let \( V = [v_1, \ldots, v_n] \)
\( \Lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \)

then \( A = V \Lambda V^{-1} \) (if \( V^{-1} \) exists)

\[ \Rightarrow \quad \dot{z} = \Lambda z, \quad z = V^{-1} x \]
\[ z(t) = e^{\Lambda t} z(0) \]

\[ \Rightarrow \quad x(t) = Ve^{\Lambda t} V^{-1} x(0) = e^{At} x(0) \]

System is stable if \( \text{Re}(\lambda_i) < 0 \)

Forced response

\[ \dot{x} = Ax + Bu \]
\[ \Rightarrow \quad x(t) = \int_0^t e^{A(t-h)} Bu(h) \, dh + e^{At} x(0) \]

System is bounded-input bounded-output (BIBO) stable:
\[ \sup_{t \geq 0} \|x(t)\| \leq \gamma \sup_{t \geq 0} \|u(t)\| \]

\[ \gamma = \|B\| \int_0^\infty \|e^{At}\| \, dt, \text{ if } \text{Re}(\lambda_i) < 0 \]

Frequency response

\[ \dot{x} = Ax + Bu \]
\[ u = U(\omega)e^{j\omega t} \Rightarrow \quad x = X(\omega)e^{j\omega t} \]
\[ \Rightarrow \quad X(\omega) = (j\omega I - A)^{-1} Bu(\omega) \]
Example: step response

Mass-spring-damper system

Equation of motion:

\[ m\ddot{x} + c(\dot{x}) + k(x) = u \]

\[ c(\dot{x}) = \dot{x} \]

\[ k(x) \text{ nonlinear:} \]

Input \( u(t) \)

Response \( x(t) \)

apparent damping ratio depends on size of input step
Example: multiple equilibria

First order system: \[ \dot{x} = f(x) \]

\[ x > x_a \quad \Rightarrow \quad f(x) > 0 \quad \Rightarrow \quad x(t) \text{ increases} \]
\[ x_b < x < x_a \quad \Rightarrow \quad f(x) < 0 \quad \Rightarrow \quad x(t) \text{ decreases} \]
\[ x_c < x < x_b \quad \Rightarrow \quad f(x) > 0 \quad \Rightarrow \quad x(t) \text{ increases} \]
\[ x < x_c \quad \Rightarrow \quad f(x) < 0 \quad \Rightarrow \quad x(t) \text{ decreases} \]

- \( x_a, x_c \) are unstable equilibrium points
- \( x_b \) is a stable equilibrium point
Example: multiple equilibria

First order system: \( \dot{x} = f(x) \)

\[ x > x_a \implies f(x) > 0 \implies x(t) \text{ increases} \]
\[ x_b < x < x_a \implies f(x) < 0 \implies x(t) \text{ decreases} \]
\[ x_c < x < x_b \implies f(x) > 0 \implies x(t) \text{ increases} \]
\[ x < x_c \implies f(x) < 0 \implies x(t) \text{ decreases} \]

- \( x_a, x_c \) are unstable equilibrium points
- \( x_b \) is a stable equilibrium point
Example: limit cycle

Van der Pol oscillator:

$$\ddot{x} + (x^2 - 1) \dot{x} + x = 0$$

- Response $x(t)$ tends to a limit cycle (= trajectory forming a closed curve)
- Amplitude independent of initial conditions

Response with $x(0) = 0.05$, $\dot{x}(0) = 0.05$

State trajectories $(x(t), \dot{x}(t))$
Example: chaotic behaviour

Strange attractor
Example: chaotic behaviour

Lorenz attractor

- Simplified model of atmospheric convection:
  \[
  \begin{align*}
  \dot{x} &= \sigma(y - x) \\
  \dot{y} &= x(\rho - z) - y \\
  \dot{z} &= xy - \beta z
  \end{align*}
  \]

- State variables
  \[
  \begin{align*}
  x(t) &: \quad \text{fluid velocity} \\
  y(t) &: \quad \text{difference in temperature of ascending and descending fluid} \\
  z(t) &: \quad \text{characterizes distortion of vertical temperature profile}
  \end{align*}
  \]

- Parameters \( \sigma = 10, \beta = 8/3, \rho = \text{variable} \)
Example: chaotic behaviour

Lorenz attractor

$\rho = 28 \Rightarrow \text{“strange attractor”:}$
Example: chaotic behaviour

Lorenz attractor

sensitivity to initial conditions
Example: chaotic behaviour

Lorenz attractor

sensitivity to initial conditions

blue: \((x, y, z) = (0, 1, 1.05)\)

red: \((x, y, z) = (0, 1, 1.050001)\)
Example: chaotic behaviour

Lorenz attractor

$\rho = 99.96 \xrightarrow{\text{limit cycle}}$
Example: chaotic behaviour

Lorenz attractor

\[ \rho = 14 \rightarrow \text{convergence to a stable equilibrium:} \]
State space equations

A continuous-time nonlinear system

\[ \dot{x} = f(x, u, t) \quad x : \text{state} \]
\[ u : \text{input} \]

e.g. \( n \)th order differential equation:

\[ \frac{d^n y}{dt^n} = h(y, \ldots, \frac{d^{n-1} y}{dt^{n-1}}, u, t) \]

has state vector (one possible choice)

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
y \\
y' \\
\vdots \\
\frac{d^{n-1} y}{dt^{n-1}}
\end{bmatrix}
\]

and state space dynamics:

\[
\dot{x} = \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n
\end{bmatrix} = \begin{bmatrix}
x_2 \\
x_3 \\
\vdots \\
h(x_1, x_2, \ldots, x_n, u, t)
\end{bmatrix} = f(x, u, t)
\]
Equilibrium points

$x^*$ is an equilibrium point of system $\dot{x} = f(x)$ if (and only if):

$$x(0) = x^* \quad \text{implies} \quad x(t) = x^* \quad \forall t > 0$$

i.e. $f(x^*) = 0$

★ Consider local stability of individual equilibrium points
★ Convention: define $f$ so that $x = 0$ is equilibrium point of interest
★ Autonomous system: $\dot{x} = f(x) \quad \Longrightarrow \quad x^* = \text{constant}$

Examples:

(a). $\ddot{y} + \alpha \dot{y}^2 + \beta \sin y = 0$  (damped pendulum)

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \quad n = 0, \pm 1$$

(b). $\ddot{y} + (y - 1)^2 \dot{y} + y - \sin(\pi y/2) = 0$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
**Equilibrium point examples**

(a). \( \dot{y} + \alpha y^2 + \beta \sin y = 0 \)

State: \( x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \), \( \dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} y \\ -\alpha y^2 - \beta \sin y \end{bmatrix} \)

**Exam.** \( \dot{x} = 0 \Rightarrow \{ \dot{y} = 0 \} \Rightarrow x^* = \begin{bmatrix} n \pi \\ 0 \end{bmatrix}, n = 0, \pm 1, \ldots \)

(b). \( \ddot{y} + (y-1)^2 \dot{y} + y - \sin(\pi y/2) = 0 \)

State: \( x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \), \( \dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \ddot{y} \\ -\ddot{y} + (y-1)^2 \dot{y} + y + \sin(\pi y/2) \end{bmatrix} \)

**Exam.** \( \dot{x} = 0 \Rightarrow \{ \dot{y} = 0 \} \Rightarrow x^* = \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} \)
An equilibrium point $x = 0$ is stable iff:

$$\max_t \|x(t)\| \text{ can be made arbitrarily small}$$

by making $\|x(0)\|$ small enough

$$\Updownarrow$$

for any $R > 0$, there exists $r > 0$ so that

$$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > 0$$

---

Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?

No guarantee of convergence to the equilibrium point.

An equilibrium point $x = 0$ is stable iff:

$$\max_t \|x(t)\| \text{ can be made arbitrarily small}$$

by making $\|x(0)\|$ small enough

\[ \Downarrow \]

for any $R > 0$, there exists $r > 0$ so that

$$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > 0$$

- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
- No guarantee of convergence to the equilibrium point
Asymptotic stability definition

An equilibrium point $x = 0$ is **asymptotically** stable iff:

(i). $x = 0$ is stable

(ii). $\|x(0)\| < r \implies \|x(t)\| \to 0$ as $t \to \infty$

(ii) is equivalent to:

for any $R > 0$,

$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > T$

for some $r$, $T$
Exponential stability definition

An equilibrium point $x = 0$ is **exponentially** stable iff:

$$||x(0)|| < r \implies ||x(t)|| \leq \beta e^{-\alpha t} \quad \forall t > 0$$

exponential stability is a special case of asymptotic stability
The region of attraction of $x = 0$ is the set of all initial conditions $x(0)$ for which $x(t) \to 0$ as $t \to \infty$.

Every asymptotically stable equilibrium point has a region of attraction.

- $r = \infty \implies$ entire state space is a region of attraction
- $\implies x = 0$ is globally asymptotically stable

- Are stable linear systems asymptotically stable?
Nonlinear state space equations:  $\dot{x} = f(x, u)$

$x = \text{state vector}$,  $u = \text{control input}$

**Equilibrium points:** $x^*$ is an equilibrium point

of $\dot{x} = f(x)$ if $f(x^*) = 0$

**Stable equilibrium point:** $x^*$ is stable if state trajectories starting close to $x^*$ remain near $x^*$ at all times

**Asymptotically stable equilibrium point:** $x^*$ must be stable and state trajectories starting near $x^*$ must tend to $x^*$ asymptotically

**Region of attraction:** the set of initial conditions from which state trajectories converge asymptotically to equilibrium $x^*$
Lecture 2

Linearization and Lyapunov’s direct method
Linearization and Lyapunov’s direct method

- Review of stability definitions
- Linearization method
- Direct method for stability
- Direct method for asymptotic stability
- Linearization method revisited
System: \( \dot{x} = f(x) \)  
- unforced system (i.e. closed-loop)
- consider stability of individual equilibrium points

0 is a **stable** equilibrium if:

\[
\|x(0)\| \leq r \implies \|x(t)\| \leq R
\]

for any \( R > 0 \)

0 is asymptotically stable if:

\[
\|x(0)\| \leq r \implies \|x(t)\| \to 0 \quad \text{as} \quad t \to \infty
\]

**Stability** → local property

**Asymptotic stability** → global if \( r = \infty \) allowed
Review of stability definitions

System: \( \dot{x} = f(x) \)  
- unforced system (i.e. closed-loop)
- consider stability of individual equilibrium points

0 is a **stable** equilibrium if:

\[
\| x(0) \| \leq r \implies \| x(t) \| \leq R \\
\text{for any } R > 0
\]

0 is **asymptotically stable** if:

\[
\| x(0) \| \leq r \implies \| x(t) \| \to 0 \\
as \ t \to \infty
\]

Stability → local property
Asymptotic stability → global if \( r = \infty \) allowed
Historical development of Stability Theory

- Potential energy in conservative mechanics (Lagrange 1788):
  An equilibrium point of a conservative system is stable if it corresponds to a minimum of the potential energy stored in the system

- Energy storage analogy for general ODEs (Lyapunov 1892)

- Invariant sets (Lefschetz, La Salle 1960s)

Lyapunov’s linearization method

- Determine stability of equilibrium at $x = 0$ by analyzing the stability of the linearized system at $x = 0$.

- **Jacobian** linearization:

\[
\dot{x} = f(x) = f(0) + \left. \frac{\partial f}{\partial x} \right|_{x=0} x + R_1(x) \\
\approx Ax
\]

where

\[
A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
\]

Jacobian matrix

\[
R_1(x) \to 0 \quad \text{as} \quad x \to 0
\]

remainder
Taylor’s Theorem reminder

- If \( f(x) \) is differentiable at \( x = 0 \), then

\[
 f(x) = f(0) + \frac{\partial f}{\partial x} \bigg|_{x=0} x + R_1(x)
\]

where \( R_1(x) \to 0 \) as \( x \to 0 \)

- More precisely: \( \lim_{x \to 0} \frac{R_1(x)}{\|x\|} = 0 \)

which implies that, for any \( \varepsilon > 0 \), there exists an \( \delta > 0 \) such that

\[
\|R_1(x)\| \leq \varepsilon \|x\| \quad \text{if} \quad \|x\| \leq \delta
\]
Lyapunov’s linearization method

Conditions on $A$ for stability of original nonlinear system at $x = 0$:

<table>
<thead>
<tr>
<th>stability of linearization</th>
<th>stability of nonlinear system at $x = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Re}(\lambda(A)) &lt; 0$</td>
<td>asymptotically stable (locally)</td>
</tr>
<tr>
<td>$\max \text{Re}(\lambda(A)) = 0$</td>
<td>stable or unstable</td>
</tr>
<tr>
<td>$\max \text{Re}(\lambda(A)) &gt; 0$</td>
<td>unstable</td>
</tr>
</tbody>
</table>
Lyapunov’s linearization method: examples

\[ x' = x \cos x \quad \Rightarrow \quad x' \approx \left( \cos x - x \sin x \right)_{x=0} \cdot x = x \]

\[ 0 \leq x = x \left( 1 - \frac{x^2}{2} + \ldots \right) \approx x \]

LINEARISATION: \( \dot{x} = x \quad \Rightarrow \quad \lambda = 1 \quad \Rightarrow \quad x = 0 \) is an **unstable** ERM

\[ \ddot{x} + \dot{x} e^x = 0 \quad \Rightarrow \quad \ddot{x} + \dot{x} \left( 1 + x + \frac{x^2}{2} + \ldots \right) = 0 \]

\[ \Rightarrow \quad \ddot{x} + \dot{x} + h.o.t. = 0 \]

\[ \Rightarrow \quad \ddot{x} + \dot{x} \leq 0 \]

LINEARISATION: \( \ddot{x} + \dot{x} = 0 \quad \Rightarrow \quad \lambda = -1, 0 \quad \Rightarrow \quad \text{INCONCLUSIVE} \]
Lyapunov’s linearization method

- Linearization may not provide enough information:
  - (stable) \( \dot{x} = -x^3 \) linearize \( \dot{x} = 0 \) (indeterminate)
  - (unstable) \( \dot{x} = x^3 \) linearize \( \dot{x} = 0 \) (indeterminate)

  ↑
  higher order terms determine stability

- Why does linear control work?
  1. Linearize the model:
     \[
     \dot{x} = f(x, u) \\
     \approx Ax + Bu, \quad A = \frac{\partial f}{\partial x}(0, 0), \quad B = \frac{\partial f}{\partial u}(0, 0)
     \]
  2. Design a linear feedback controller using the linearized model:
     \[
     u = -Kx, \quad \max \text{Re}(\lambda(A - BK)) < 0
     \]
     closed-loop linear model strictly stable
     nonlinear system \( \dot{x} = f(x, -Kx) \) is locally asymptotically stable at \( x = 0 \)
Lyapunov’s linearization method

- Linearization may not provide enough information:

  (stable) \[ \dot{x} = -x^3 \xrightarrow{\text{linearize}} \dot{x} = 0 \] (indeterminate)

  (unstable) \[ \dot{x} = x^3 \xrightarrow{\text{linearize}} \dot{x} = 0 \] (indeterminate)

  \[ \uparrow \]

  higher order terms determine stability

- Why does linear control work?

  1. Linearize the model:

     \[ \dot{x} = f(x, u) \]

     \[ \approx Ax + Bu, \quad A = \frac{\partial f}{\partial x}(0, 0), \quad B = \frac{\partial f}{\partial u}(0, 0) \]

  2. Design a linear feedback controller using the linearized model:

     \[ u = -Kx, \quad \max \text{Re}(\lambda(A - BK)) < 0 \]

     closed-loop linear model strictly stable

     nonlinear system \[ \dot{x} = f(x, -Kx) \] is locally asymptotically stable at \( x = 0 \)
Lyapunov’s direct method: mass-spring-damper example

Equation of motion: \[ m\ddot{y} + c(\dot{y}) + k(y) = 0 \]

Stored energy: \[ V = \text{K.E.} + \text{P.E.} \]
\[ \begin{align*}
\text{K.E.} &= \frac{1}{2}m\dot{y}^2 \\
\text{P.E.} &= \int_0^y k(y) \, dy
\end{align*} \]

Rate of energy dissipation \[ \dot{V} = \frac{1}{2}m\dot{y} \frac{d}{dy} \dot{y}^2 + \dot{y} \frac{d}{dy} \left[ \int_0^y k(y) \, dy \right] \]
\[ = m\ddot{y} + \dot{y}k(y) \]

but \( m\ddot{y} + k(y) = -c(\dot{y}) \), so \[ \dot{V} = -c(\dot{y})\dot{y} \leq 0 \quad \leftarrow \text{since } \text{sign}(c(\dot{y})) = \text{sign}(\dot{y}) \]
Lyapunov’s direct method: mass-spring-damper example

Equation of motion: \[ m\ddot{y} + c(\dot{y}) + k(y) = 0 \]

Stored energy: \[ V = K.E. + P.E. \]
\[ \{ \begin{align*}
K.E. &= \frac{1}{2}m\dot{y}^2 \\
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\end{align*} \]

Rate of energy dissipation

\[ \dot{V} = \frac{1}{2}m\dot{y} \frac{d}{dy} \dot{y}^2 + \dot{y} \frac{d}{dy} \left[ \int_0^y k(y) \, dy \right] \\
= m\ddot{y} + \dot{y}k(y) \]

but \( m\ddot{y} + k(y) = -c(\dot{y}) \), so

\[ \dot{V} = -c(\dot{y})\dot{y} \leq 0 \]
\[ \text{since } \text{sign}(c(\dot{y})) = \text{sign}(\dot{y}) \]
System state: e.g. \( x = [y \ y]^T \)

\( \dot{V}(x) \leq 0 \) implies that \( x = 0 \) is stable

\[ V(x(t)) \text{ must decrease over time} \]

but

\[ V(x) \text{ increases with increasing } \|x\| \]

Formal argument:

for any given \( R > 0 \):

\[
\|x\| < R \quad \text{whenever} \quad V(x) < \overline{V} \quad \text{for some } \overline{V}
\]

and \( V(x) < \overline{V} \) whenever \( \|x\| < r \) for some \( r \)

\[ \therefore \|x(0)\| < r \quad \implies \quad V(x(0)) < \overline{V} \]

\[ \implies \quad V(x(t)) < \overline{V} \quad \text{for all } t > 0 \]

\[ \implies \quad \|x(t)\| < R \quad \text{for all } t > 0 \]
Mass-spring-damper example contd.

- System state: e.g. \( x = [y \ y]^T \)
- \( \dot{V}(x) \leq 0 \) implies that \( x = 0 \) is stable
  \[ \uparrow \]
  \[ V(x(t)) \] must decrease over time
  but
  \[ V(x) \] increases with increasing \( \|x\| \)

- Formal argument:
  for any given \( R > 0 \):
  \[ \|x\| < R \quad \text{whenever} \quad V(x) < \bar{V} \quad \text{for some} \quad \bar{V} \]
  and \( V(x) < \bar{V} \quad \text{whenever} \quad \|x\| < r \quad \text{for some} \quad r \)

  \[ \therefore \quad \|x(0)\| < r \quad \implies \quad V(x(0)) < \bar{V} \]
  \[ \implies \quad V(x(t)) < \bar{V} \quad \text{for all} \quad t > 0 \]
  \[ \implies \quad \|x(t)\| < R \quad \text{for all} \quad t > 0 \]
Positive definite functions

- What if $V(x)$ is not monotonically increasing in $\|x\|$?
- Same arguments apply if $V(x)$ is continuous and positive definite, i.e.

  (i). $V(0) = 0$
  (ii). $V(x) > 0$ for all $x \neq 0$

for any given $\overline{V} > 0$, can always find $r$ so that

$V(x) < \overline{V}$ whenever $\|x\| < r$

$V(x) \geq \alpha \|x\|^n$

for some constants $\alpha$, $n$, so

$\|x\| < (\overline{V}/\alpha)^{1/n}$ whenever $V(x) < \overline{V}$
Positive definite functions

- What if \( V(x) \) is not monotonically increasing in \( \|x\| \)?
- Same arguments apply if \( V(x) \) is continuous and positive definite, i.e.

\[
\begin{align*}
\text{(i). } & V(0) = 0 \\
\text{(ii). } & V(x) > 0 \quad \text{for all } x \neq 0
\end{align*}
\]

for any given \( V > 0 \), can always find \( r \) so that

\[ V(x) < V \text{ whenever } \|x\| < r \]

\[ V(x) \geq \alpha \|x\|^n \]

for some constants \( \alpha, n \), so

\[ \|x\| < (V/\alpha)^{1/n} \quad \text{whenever } \quad V(x) < V \]
Lyapunov stability theorem

If there exists a continuous function $V(x)$ such that

- $V(x)$ is positive definite
- $\dot{V}(x) \leq 0$

then $x = 0$ is stable.

To show that this implies $\|x(t)\| < R$ for all $t > 0$ whenever $\|x(0)\| < r$ for any $R$ and some $r$:

1. choose $\bar{V}$ as the minimum of $V(x)$ for $\|x\| = R$

2. find $r$ so that $V(x) < \bar{V}$ whenever $\|x\| < r$

3. then $\dot{V}(x) \leq 0$ ensures that

$$V(x(t)) < \bar{V} \quad \forall t > 0 \quad \text{if} \quad \|x(0)\| < r$$

$$\therefore \quad \|x(t)\| < R \quad \forall t > 0$$
Lyapunov stability theorem

If there exists a continuous function $V(x)$ such that

- $V(x)$ is positive definite
- $\dot{V}(x) \leq 0$

then $x = 0$ is stable.

To show that this implies $\|x(t)\| < R$ for all $t > 0$ whenever $\|x(0)\| < r$ for any $R$ and some $r$:

1. choose $\overline{V}$ as the minimum of $V(x)$ for $\|x\| = R$

2. find $r$ so that $V(x) < \overline{V}$ whenever $\|x\| < r$

3. then $\dot{V}(x) \leq 0$ ensures that

$$V(x(t)) < \overline{V} \quad \forall t > 0 \quad \text{if} \quad \|x(0)\| < r$$

$$\therefore \|x(t)\| < R \quad \forall t > 0$$
Lyapunov stability theorem

- Lyapunov’s direct method also applies if $V(x)$ is locally positive definite, i.e. if

(i). $V(0) = 0$
(ii). $V(x) > 0$ for $x \neq 0$ and $\|x\| < R_0$

then $x = 0$ is stable if $\dot{V}(x) \leq 0$ whenever $\|x\| < R_0$.

- Apply the theorem without determining $R, r$
  - only need to find p.d. $V(x)$ satisfying $\dot{V}(x) \leq 0$.

- Examples

(i). $\dot{x} = -a(t)x, \quad a(t) > 0$

$$V = \frac{1}{2}x^2 \implies \dot{V} = x\dot{x} = -a(t)x^2 \leq 0$$

(ii). $\dot{x} = -a(x), \quad \text{sign}(a(x)) = \text{sign}(x)$

$$V = \frac{1}{2}x^2 \implies \dot{V} = x\dot{x} = -a(x)x \leq 0$$
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(i). $\dot{x} = -a(t)x, \quad a(t) > 0$

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  $V = \frac{1}{2}x^2 \implies \dot{V} = x\dot{x} = -a(x)x \leq 0$
Lyapunov stability theorem

More examples

(iii). \[ \dot{x} = -a(x), \quad \int_0^x a(x) \, dx > 0 \]

\[ V = \int_0^x a(x) \, dx \quad \Rightarrow \quad \dot{V} = a(x) \dot{x} \]

\[ = -a^2(x) \leq 0 \]

(iv). \[ \ddot{\theta} + \sin \theta = 0 \]

\[ V = \frac{1}{2} \dot{\theta}^2 + \int_0^\theta \sin \theta \, d\theta \quad \Rightarrow \quad \dot{V} = \ddot{\theta} \dot{\theta} + \dot{\theta} \sin \theta \]

\[ = 0 \]
Asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

\[
\begin{align*}
V(x) & \quad \text{is positive definite} \\
\dot{V}(x) & \quad \text{is negative definite}
\end{align*}
\]

then $x = 0$ is \textbf{locally asymptotically stable}.

($\dot{V}$ negative definite $\iff -\dot{V}$ positive definite)

Asymptotic convergence $x(t) \to 0$ as $t \to \infty$ can be shown by contradiction:

if $\|x(t)\| > R'$ for all $t \geq 0$, then

\[
\begin{align*}
\dot{V}(x) & < -W \\
V(x) & \geq V
\end{align*}
\]

for all $t \geq 0$

\[
\uparrow \quad \text{contradiction}
\]
Asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

\[
\begin{align*}
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\]

then $x = 0$ is locally asymptotically stable.

$(\dot{V} \text{ negative definite} \iff -\dot{V} \text{ positive definite})$

Asymptotic convergence $x(t) \to 0$ as $t \to \infty$ can be shown by contradiction:

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\[
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\]

for all $t \geq 0$

contradiction
Asymptotic stability result also applies if $\dot{V}(x)$ is only \textit{locally} negative definite.

Why does the linearization method work?

- consider 1st order system: $\dot{x} = f(x)$
  linearize about $x = 0$: $\dot{x} = -ax + R(x)$
- assume $a > 0$ and try Lyapunov function $V$:
  
  \[
  V(x) = \frac{1}{2}x^2 \\
  \dot{V}(x) = x\dot{x} = -ax^2 + xR(x) = -x^2\left(a - \frac{R(x)}{x}\right) \\
  \leq -x^2\left(a - \frac{|R(x)|}{x}\right)
  \]
- but we can choose $\epsilon$ so that $|R(x)/x| < \epsilon$ whenever $|x| \leq r$, so
  
  \[
  \dot{V} \leq -x^2(a - \epsilon) \\
  \leq -\gamma x^2 \quad \text{with} \quad a - \epsilon = \gamma > 0 \text{ if } |x| \leq r
  \]
  
  $\Rightarrow \dot{V}$ negative definite for $|x|$ small enough
  $\Rightarrow x = 0$ locally asymptotically stable
  
  Generalization to $n$th order systems is straightforward
Linearization method and asymptotic stability

- Asymptotic stability result also applies if $\dot{V}(x)$ is only locally negative definite.

- Why does the linearization method work?
  - consider 1st order system: $\dot{x} = f(x)$
    linearize about $x = 0$: $\dot{x} = -ax + R(x)$
  - assume $a > 0$ and try Lyapunov function $V$:
    $V(x) = \frac{1}{2}x^2$
    $\dot{V}(x) = x\dot{x} = -ax^2 + xR(x) = -x^2(a - R(x)/x)$
    $\leq -x^2(a - |R(x)/x|)$
  - but we can choose $\epsilon$ so that $|R(x)/x| < \epsilon$ whenever $|x| \leq r$, so
    $\dot{V} \leq -x^2(a - \epsilon)$
    $\leq -\gamma x^2$ with $a - \epsilon = \gamma > 0$ if $|x| \leq r$
  - $\Rightarrow \dot{V}$ negative definite for $|x|$ small enough
  - $\Rightarrow x = 0$ locally asymptotically stable

Generalization to $n$th order systems is straightforward
Global asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

\[
\begin{align*}
V(x) & \quad \text{is positive definite} \\
\dot{V}(x) & \quad \text{is negative definite} \\
V(x) & \rightarrow \infty \text{ as } \|x\| \rightarrow \infty
\end{align*}
\]

for all $x$,

then $x = 0$ is globally asymptotically stable

- If $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, then $V(x)$ is radially unbounded

- Test whether $V(x)$ is radially unbounded by checking if $V(x) \rightarrow \infty$ as each individual element of $x$ tends to infinity (necessary).
Global asymptotic stability theorem

- Global asymptotic stability requires:
  \[ \|x(t)\| \text{ finite} \]
  \[ \begin{cases} 
  \text{for all } t > 0 \\
  \text{for all } x(0) 
  \end{cases} \]
  \[ \uparrow \]
  not guaranteed by \( \dot{V} \) negative definite
  in addition to asymptotic stability of \( x = 0 \)

- Hence add extra condition: \( V(x) \to \infty \) as \( \|x\| \to \infty \)
  \[ \uparrow \text{ equiv. to} \]
  level sets \( \{ x : V(x) = V \} \) are bounded
  \[ \uparrow \text{ equiv. to} \]
  \[ \|x\| \text{ is finite whenever } V(x) \text{ is finite} \]
  \[ \uparrow \]
  prevents \( x(t) \) drifting away from 0 despite \( \dot{V} < 0 \)
Asymptotic stability example

System:
\[
\begin{align*}
\dot{x}_1 &= (x_2 - 1)x_1^3 \\
\dot{x}_2 &= -\frac{x_2}{(1 + x_1^2)^2} - \frac{x_2}{1 + x_2^2}
\end{align*}
\]

- Trial Lyapunov function \( V(x) = x_1^2 + x_2^2 \):
\[
\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2
\]
\[
= -2x_1^4 + 2x_2x_1^4 - 2\frac{x_2x_1^4}{(1 + x_1^2)^2} - 2\frac{x_2^2}{1 + x_2^2} \leq 0
\]

\[\uparrow\]
change \( V \) to make these terms cancel
Asymptotic stability example

- New trial Lyapunov function $V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$:

  $\dot{V}(x) = 2\left[\frac{x_1}{1 + x_1^2} - \frac{x_1^3}{(1 + x_1^2)^2}\right]x_1 + 2x_2 x_2$

  $= -2\frac{x_1^4}{(1 + x_1^2)^2} - 2\frac{x_2^2}{1 + x_2^2} \leq 0$

$V(x)$ positive definite, $\dot{V}(x)$ negative definite $\implies x = 0$ a.s.

But $V(x)$ not radially unbounded, so cannot conclude global asymptotic stability

State trajectories:
Summary

- Positive definite functions
- Derivative of $V(x)$ along trajectories of $\dot{x} = f(x)$
- Lyapunov’s direct method for: stability, asymptotic stability, global stability
- Lyapunov’s linearization method
Lecture 3

Convergence and invariant sets
Convergence and invariant sets

- Review of Lyapunov’s direct method
- Convergence analysis using Barbalat’s Lemma
- Invariant sets
- Global and local invariant set theorems
Review of Lyapunov’s direct method

Positive definite functions

- If

\[
V(0) = 0 \\
V(x) > 0 \quad \text{for all } x \neq 0
\]

then \(V(x)\) is positive definite

- If \(S\) is a set containing \(x = 0\) and

\[
V(0) = 0 \\
V(x) > 0 \quad \text{for all } x \neq 0, x \in S
\]

then \(V(x)\) is locally positive definite (within \(S\))

- e.g.

\[
V(x) = x^\top x \quad \leftarrow \quad \text{positive definite}
\]

\[
V(x) = x^\top x(1 - x^\top x) \quad \leftarrow \quad \text{locally positive definite within } S = \{x : x^\top x < 1\}
System: \( \dot{x} = f(x), \quad f(0) = 0 \)

Storage function: \( V(x) \)

Time-derivative of \( V \): \( \dot{V}(x) = \frac{\partial V}{\partial x} \frac{dx}{dt} = \nabla V(x)^\top \dot{x} = \nabla V(x)^\top f(x) \)

- If
  
  (i). \( V(x) \) is positive definite
  (ii). \( \dot{V}(x) \leq 0 \)

  \{ \text{for all } x \in S \}

then the equilibrium \( x = 0 \) is stable

- If

  (iii). \( \dot{V}(x) \) is negative definite \( \text{for all } x \in S \)

then the equilibrium \( x = 0 \) is asymptotically stable

- If

  (iv). \( S = \) entire state space
  (v). \( V(x) \to \infty \text{ as } \|x\| \to \infty \)

then the equilibrium \( x = 0 \) is globally asymptotically stable
Convergence analysis

What can be said about convergence of $x(t)$ to 0 if $\dot{V}(x) \leq 0$ but $\dot{V}(x)$ is not negative definite?

Revisit m-s-d example:

Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$

Storage function: $V = K.E. + P.E. = \frac{1}{2}m\dot{y}^2 + \int_0^y k(y) \, dy$

$\dot{V} = -c(\dot{y})\dot{y}$
Convergence analysis

- $V$ is p.d. and $\dot{V} \leq 0$ so: $(y, \dot{y}) = (0, 0)$ is stable
  and $V(y, \dot{y})$ tends to a finite limit as $t \to \infty$

- but does $(y, \dot{y})$ converge to $(0, 0)$?

\[ \uparrow \text{ equivalent to} \]

- can $V(y, \dot{y})$ “get stuck” at $V = V_0 \neq 0$ as $t \to \infty$?

\[ \downarrow \]

- need to consider motion at points $(y, \dot{y})$ for which $\dot{V} = 0$
Example

Equation of motion: \( m\ddot{y} + c(\dot{y}) + k(y) = 0 \)

- \( k(y) = 5 \tan^{-1}(y/5) \)
- \( c(\dot{y}) = 0.1\dot{y}(0.5 + |\dot{y}|)(2 - e^{-0.1|\dot{y}|}) \)

Storage function:
\[
V = \frac{1}{2} \dot{y}^2 + \int_0^y 5 \tan^{-1}(y/5) \, dy
\]
\[
\dot{V} = -c(\dot{y})\dot{y} \leq 0
\]
\[\downarrow \]
\[\dot{V} = 0 \text{ when } \dot{y} = 0 \]

but \( k(y) \neq 0 \implies \dot{y} \neq 0 \implies \ddot{V} \neq 0 \)
\[\downarrow \]

\( V \) continues to decrease until \( y = \dot{y} = 0 \)
Example

Equation of motion: \( m\ddot{y} + c(\dot{y}) + k(y) = 0 \)

\[ k(y) = 5 \tan^{-1}(y/5) \]

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Storage function:

\[ V = \frac{1}{2} \dot{y}^2 + \int_0^y 5 \tan^{-1}(y/5) \, dy \]

\[ \dot{V} = -c(\dot{y})\dot{y} \leq 0 \]

\[ \dot{V} = 0 \text{ when } \dot{y} = 0 \]

but \( k(y) \neq 0 \implies \ddot{y} \neq 0 \implies \dot{V} \neq 0 \)

\[ V \text{ continues to decrease until } y = \dot{y} = 0 \]
Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \rightarrow 0$ as $t \rightarrow \infty$

2. determine the set $\mathcal{R}$ of points $x$ for which $\dot{V}(x) = 0$

3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times

then $x(t)$ has to converge to $\mathcal{M}$ as $t \rightarrow \infty$

This approach is the basis of the invariant set theorems
Barbalat’s Lemma

Barbalat’s lemma: For any function $\phi(t)$, if

1. $\int_0^t \phi(\tau)\,d\tau$ converges to a finite limit as $t \to \infty$
2. $\dot{\phi}(t)$ exists and remains finite for all $t$

then $\lim_{t \to \infty} \phi(t) = 0$

- If $\phi$ is uniformly continuous, then
  $$\int_0^t \phi(\tau)\,d\tau \to \text{constant} \implies \phi(t) \to 0 \text{ as } t \to \infty$$

- Condition (ii) ensures that $\phi(t)$ is continuous for all $t$

- Without (ii) we could have $\int_0^t \phi(\tau)\,d\tau \to \text{constant}$ and $\phi(t) \not\to 0$ as $t \to \infty$
Barbalat’s Lemma

Example: pulse train \( \phi(t) = \sum_{k=0}^{\infty} e^{-4k(t-k)^2} \):

\[
\phi(t) = \int_0^t \phi(\tau) \, d\tau:
\]

From the plots it is clear that

\[
\int_0^t \phi(s) \, ds \text{ tends to a finite limit}
\]

but \( \phi(t) \not\to 0 \) as \( t \to \infty \) because \( \dot{\phi}(t) \to \infty \) as \( t \to \infty \)
Barbalat’s Lemma

Apply Barbalat’s Lemma to $\dot{V}(x(t)) = \phi(t) \leq 0$:

(a) Integrate:

$$\int_0^t \phi(s) \, ds = V(x(t)) - V(x(0))$$

$\leftarrow$ finite limit as $t \to \infty$

(b) Differentiate:

$$\dot{\phi}(t) = \dot{V}(x(t)) = f(x)^\top \frac{\partial^2 V}{\partial x^2}(x)f(x) + \nabla V(x)^\top \frac{\partial f}{\partial x}(x)f(x)$$

$= \text{finite for all } t \text{ if } f(x) \text{ continuous and } V(x) \text{ continuously differentiable}$

$\Downarrow$

$$\dot{V}(x) \to 0 \text{ as } t \to \infty$$

(a) and (b) rely on $\|x(t)\|$ remaining finite for all $t$, which is implied by:

$V(x)$ positive definite

$\dot{V}(x) \leq 0$

$V(x) \to \infty \text{ as } \|x\| \to \infty$
Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
   → true whenever $\dot{V} \leq 0$ & $V, f$ are smooth & $\|x(t)\|$ is bounded
   [by Barbalat’s Lemma]

2. determine the set $\mathcal{R}$ of points $x$ for which $\dot{V}(x) = 0$
   → algebra!

3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times
   → $\mathcal{M}$ must be invariant

   then $x(t)$ has to converge to $\mathcal{M}$ as $t \to \infty$

This approach is the basis of the invariant set theorems
Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
   
   $\to$ true whenever $\dot{V} \leq 0$ & $V, f$ are smooth & $\|x(t)\|$ is bounded

   [by Barbalat’s Lemma]

2. determine the set $\mathcal{R}$ of points $x$ for which $\dot{V}(x) = 0$
   
   $\to$ algebra!

3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times
   
   $\to$ $\mathcal{M}$ must be invariant

   then $x(t)$ has to converge to $\mathcal{M}$ as $t \to \infty$

This approach is the basis of the invariant set theorems
Invariant sets

- A set of points $\mathcal{M}$ in state space is **invariant** if

$$x(t_0) \in \mathcal{M} \implies x(t) \in \mathcal{M} \quad \text{for all } t > t_0$$

**Examples:**

- Equilibrium points
- Limit cycles
- If $\dot{V}(x) \leq 0$, then sublevel sets of $V(x)$ are invariant

$$\text{sublevel sets of } V(x) \text{ are } \{ x : V(x) \leq \alpha \} \text{ for constant } \alpha$$

- If $\dot{V}(x) \to 0$ as $t \to \infty$, then

$$x(t) \text{ must converge to an invariant set } \mathcal{M} \text{ contained within the set of points on which } \dot{V}(x) = 0$$

as $t \to \infty$
Global invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

\[
V(x) \text{ is positive definite} \\
\dot{V}(x) \leq 0 \\
V(x) \to \infty \text{ as } \|x\| \to \infty
\]

then:

(i). $\dot{V}(x) \to 0$ as $t \to \infty$

(ii). $x(t) \to \mathcal{M} =$ the largest invariant set contained in $\mathcal{R}$

where $\mathcal{R} = \{x : \dot{V}(x) = 0\}$

- $\dot{V}(x)$ negative definite $\implies \mathcal{M} = 0$ \hspace{1cm} (c.f. Lyapunov’s direct method)
- Determine $\mathcal{M}$ by considering system dynamics within $\mathcal{R}$
Global invariant set theorem

Revisit m-s-d example

- $V(x)$ is positive definite, $V(x) \to \infty$ as $\|x\| \to \infty$, and
  $$\dot{V}(y, \dot{y}) = -c(\dot{y})\dot{y} \leq 0$$

- therefore $\dot{V} \to 0$, implying $\dot{y} \to 0$ as $t \to \infty$
  i.e. $\mathcal{R} = \{(y, \dot{y}) : \dot{y} = 0\}$

- but $\dot{y} = 0$ implies $\ddot{y} = -k(y)/m$

- therefore $\ddot{y} \neq 0$ unless $y = 0$, so $\dot{y}(t) = 0$ for all $t$ only if $y(t) = 0$
  i.e. $\mathcal{M} = \{(y, \dot{y}) : (y, \dot{y}) = (0, 0)\}$

\[ \Downarrow \]

$(y, \dot{y}) = (0, 0)$ is a **globally asymptotically stable equilibrium**!
Local invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

the sublevel set $\Omega = \{x : V(x) \leq \alpha\}$ is bounded for some $\alpha$
and $\dot{V}(x) \leq 0$ whenever $x \in \Omega$

then:

(i). $\Omega$ is an invariant set
(ii). $x(0) \in \Omega \implies \dot{V}(x) \to 0$ as $t \to \infty$
(iii). $x(t) \to \mathcal{M} =$ largest invariant set contained in $\mathcal{R} \cap \Omega$

where $\mathcal{R} = \{x : \dot{V}(x) = 0\}$
Local invariant set theorem

- $V(x)$ doesn’t have to be positive definite or radially unbounded

- Result is based on Barbalat’s Lemma applied to $\dot{V}$

  applies here because boundedness of $\Omega$ implies $\|x(t)\|$ finite for all $t$
  since $x(0) \in \Omega$ and $\dot{V} \leq 0$

- $\Omega$ is a region of attraction for $\mathcal{M}$
Example: local invariant set theorem

- Second order system:
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -(x_1 - 1)^2 x_2^3 - x_1 + \sin(\pi x_1 / 2)
  \end{align*}
  \]

- Equilibrium points: \((x_1, x_2) = (0, 0), (1, 0), (-1, 0)\)

- Trial storage function:
  \[
  V(x) = \frac{1}{2} x_2^2 + \int_0^{x_1} (y - \sin(\pi y / 2)) \, dy
  \]

\(V\) is not positive definite
but \(V(x) \to \infty\) if \(x_1 \to \infty\) or \(x_2 \to \infty\)

\[
\begin{align*}
&\downarrow \\
\text{sublevel sets of } V \text{ are bounded}
\end{align*}
\]
Example: local invariant set theorem

- Differentiate: \[ \dot{V}(x) = -(x_1 - 1)^2 x_2^4 \leq 0 \]

\[ \dot{V}(x) = 0 \iff x \in \mathcal{R} = \{ x : x_1 = 1 \text{ or } x_2 = 0 \} \]

- From the system model, \( x \in \mathcal{R} \) implies:

\[
\begin{align*}
x_1 &= 1 \implies (\dot{x}_1, \dot{x}_2) = (x_2, 0) \\
\text{and} \\
x_2 &= 0 \implies (\dot{x}_1, \dot{x}_2) = (0, \sin(\pi x_1/2) - x_1)
\end{align*}
\]

therefore \[
\left\{ \begin{array}{l}
x(t) \text{ remains on line } x_1 = 1 \text{ only if } x_2 = 0 \\
x(t) \text{ remains on line } x_2 = 0 \text{ only if } x_1 = 0, 1 \text{ or } -1
\end{array} \right.
\]

\[ \implies \mathcal{M} = \{(0, 0), (1, 0), (-1, 0)\} \]
Example: local invariant set theorem

System: \( \dot{x}_1 = x_2 \)

\[ \dot{x}_2 = - (x_1 - 1)^2 x_2^3 - x_1 + \sin(\pi x_1/2) \]

\[ \begin{cases} x_1 = 1 \\ x_1 = x_2 = 0 \quad \text{iff} \quad x_2 = 0 \end{cases} \]

\[ \begin{cases} x_2 = 0 \\ \dot{x}_2 = -x_1 + \sin(\pi x_1/2) = 0 \quad \text{iff} \quad x_1 = 0, \pm 1 \]
Apply the local invariant set theorem to any sublevel set
\( \Omega = \{ x : V(x) \leq \alpha \} \) containing \( x(0) \):

\[
\Omega \text{ is bounded} \quad \dot{V} \leq 0
\]

\[ \implies \quad x(t) \to M = \{(0,0), (1,0), (-1,0)\} \text{ as } t \to \infty
\]

For any given \( x(0) \), we can choose sufficiently large \( \alpha \) so that
\( \Omega = \{ x : V(x) \leq \alpha \} \) contains \( x(0) \)

so \( x(t) \to M = \{(0,0), (1,0), (-1,0)\} \) as \( t \to \infty \) for all \( x(0) \)

Can we find more precise limits for \( x(t) \)?
Example: local invariant set theorem

We have shown $x(t)$ converges asymptotically to $(0, 0)$, $(1, 0)$ or $(-1, 0)$ but:

(a). $x = (0, 0)$ is unstable since the linearization at $(0, 0)$ has poles $\pm \sqrt{\frac{\pi}{2} - 1}$

(b). $V(x)$ has sublevel sets that contain only $(1, 0)$ or $(-1, 0)$

apply the local invariant set theorem to $\Omega = \{x : V(x) \leq \alpha\}$ for $\alpha < 0$

$x = (1, 0), x = (-1, 0)$ are stable equilibrium points
Summary

- Convergence analysis using Barbalat’s lemma
- Invariant sets
- Invariant set methods for convergence analysis:
  - local invariant set theorem
  - global invariant set theorem
Lecture 4

Linear systems, passivity, and the circle criterion
Linear systems, passivity, and the circle criterion

- Summary of stability methods
- Lyapunov functions for linear systems
- Passive linear systems
- The circle criterion
Summary of stability methods

▶ Linearization method
\[ \dot{x} = Ax \text{ is strictly stable, } A = \frac{\partial f}{\partial x} \bigg|_{x=0} \]
\[ \newline \]
\[ x = 0 \text{ locally asymptotically stable} \]

▶ Lyapunov’s direct method
\[ V(x) \text{ locally p.d.} \]
\[ \dot{V}(x) \leq 0 \text{ locally} \]
\[ x = 0 \text{ stable} \]

\[ V(x) \text{ locally p.d.} \]
\[ \dot{V}(x) \text{ locally n.d.} \]
\[ x = 0 \text{ locally asymptotically stable} \]

▶ Invariant set theorems
\[ V(x) \text{ p.d.} \]
\[ \dot{V}(x) \leq 0 \]
\[ V(x) \to \infty \text{ as } \|x\| \to \infty \]
\[ \Omega = \{x : V(x) \leq V_0\} \text{ bounded} \]
\[ \dot{V}(x) \leq 0 \text{ for all } x \in \Omega \]
\[ x(t) \text{ converges to the union of invariant sets contained in } \{x : \dot{V}(x) = 0\} \]
Summary of stability methods

- **Instability theorems** analogous to Lyapunov’s direct method, e.g.

\[
V(x) \text{ p.d.}\quad \dot{V}(x) \text{ p.d.}\quad \implies \quad x = 0 \text{ unstable}
\]

- **Lyapunov stability criteria** are only **sufficient**, e.g.

\[
V(x) \text{ p.d.}\quad \dot{V}(x) \preceq 0\quad \not\iff \quad x = 0 \text{ unstable}
\]

(some other \(V(x)\) demonstrating stability may exist)

- **Converse theorems**

\[
x = 0 \text{ stable}\quad \implies \quad V(x) \text{ demonstrating stability exists}
\]

(can swap premises and conclusions in Lyapunov’s direct method)

\[\uparrow\]

But no general method for constructing \(V(x)\)
Linear systems

- **Systematic method** for constructing storage function $V(x) = x^\top Px$

  $\dot{x} = Ax$ strictly stable $\implies$ can always find constant matrix $P$ so that $\dot{V}(x)$ is negative definite

- Only need consider symmetric $P$

  $$x^\top Px = \frac{1}{2} x^\top Px + \frac{1}{2} x^\top P^\top x = \frac{1}{2} x^\top (P + P^\top) x$$

- Need $\lambda(P) > 0$ for positive definite $V(x) = x^\top Px$

  $$P = U\Lambda U^\top$$

  $\downarrow$

  $$x^\top Px = z^\top \Lambda z$$

  $\downarrow$

  $x^\top Px$ positive definite iff $\Lambda$ strictly positive

  $\downarrow$

  $z = U^\top x$

  $\left\{\text{notation: } P \succ 0\right\}$

  or “$P$ is positive definite”
Linear systems

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- Need $\lambda(P) > 0$ for positive definite $V(x) = x^\top Px$

  $$ P = U \Lambda U^\top $$
  \[
  \begin{align*}
  &\downarrow \\
  x^\top Px &= z^\top \Lambda z \\
  \downarrow \\
  x^\top Px \text{ positive definite} \quad &\iff \Lambda \text{ strictly positive} \\
  \\ &\left\{ \text{notation: } P \succ 0 \right\} \quad \text{or "} P \text{ is positive definite"}
  \end{align*}
  $$

  eigenvector/value decomposition

  $z = U^\top x$
How is $P$ computed?

\[
\begin{align*}
\dot{x} &= Ax \\
V(x) &= x^\top Px
\end{align*}
\]

\[
\implies \quad \dot{V}(x) = x^\top P \dot{x} + \dot{x}^\top Px = x^\top (PA + A^\top P)x
\]

\[
\therefore x = 0 \text{ is globally asymptotically stable if, for some } Q:
\]

\[
PA + A^\top P = -Q \\
Q = Q^\top \succ 0
\]

Lyapunov matrix equation

Pick $Q \succ 0$ and solve $PA + A^\top P = -Q$ for $P$, then

\[
\text{Re}[\lambda(A)] < 0 \iff \text{unique solution for } P \text{ and } P = P^\top \succ 0
\]
Claim: \( PA + A^T P = -Q \) has a unique solution \( P > 0 \) for every \( Q > 0 \) if and only if \( \text{Re} \left[ \lambda(A) \right] < 0 \)

Proof: Let \( \dot{x} = Ax \) and \( V = \frac{1}{2} x^T P x \)

1. If \( PA + A^T P = -Q \) with \( P, Q > 0 \), then:
   - \( V \) is positive definite
   - \( \dot{V} = \frac{1}{2} x^T (A^T P + PA) x = -\frac{1}{2} x^T Q x \) is negative definite
   - So \( \text{Re} \left[ \lambda(A) \right] < 0 \)

2. If \( Q \geq \text{Re} \left[ \lambda(A) \right] \) then \( x(t) = e^{At} x(0) \) and \( \dot{V} = -\frac{1}{2} x^T Q x \) implies
   \[
   \int_0^\infty \dot{V}(t) \, dt = -\frac{1}{2} x^T(0) \int_0^\infty e^{A^T t} Q e^{A t} \, dt \cdot x(0)
   \]
   - \( V(t) \to \lim_{t \to \infty} V(t) = x^T(0) \cdot \left( \frac{1}{2} \int_0^\infty e^{A^T t} Q e^{A t} \, dt \right) \cdot x(0) \)
   - So \( \int_0^\infty e^{A^T t} Q e^{A t} \, dt = \rho \)
   - \( V(0) - \lim_{t \to \infty} V(t) = x^T(\infty) \cdot \rho \cdot x(0) \)
   - \( \Rightarrow V = x^T P x \) and \( \rho > 0 \) if \( Q > 0 \)
Example: Lyapunov matrix equation

Stable linear system $\dot{x} = Ax$: 

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & -16 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\quad \lambda(A) = -1 \pm i\sqrt{15}
$$

Solve $PA + A^\top P = -Q$ for $P$:

$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies P_1 = \begin{bmatrix} 0.33 & -0.5 \\ -0.5 & 4.25 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 0.41 & -0.19 \\ -0.19 & 0.11 \end{bmatrix} \implies P_2 = \begin{bmatrix} 0.12 & -0.21 \\ -0.21 & 1.67 \end{bmatrix}$

\[ x^\top P_1 x = \text{constant} \]
\[ x^\top P_2 x = \text{constant} \]

* any choice of $Q \succ 0$ gives $P \succ 0$ (since $A$ is strictly stable)
* but not every $P \succ 0$ gives $Q \succ 0$
Passive systems

**Systematic method** for constructing storage functions based on the input-output representation of a system:

\[ u \rightarrow \text{system} \rightarrow y \]

- **input:** \( u \)
- **output:** \( y \)

The system mapping \( u \) to \( y \) is:

- **passive** if
  \[ \dot{V} = yu - g \quad \text{with} \quad V(t) \geq 0, \quad g(t) \geq 0 \]

- **dissipative** if it is passive and
  \[ \int_{0}^{\infty} g \, dt > 0 \quad \text{whenever} \quad \int_{0}^{\infty} yu \, dt \neq 0 \]
Passive systems

▷ Passivity is motivated by electrical networks with no internal power generation

\[
\begin{align*}
\text{input: } i & \quad \text{output: } v \\
& \quad \left\{ \begin{array}{l}
\text{stored energy: } V = \int_0^t vi \, dt \geq 0 \\
\dot{V} = iv
\end{array} \right.
\end{align*}
\]

▷ Passive mechanical systems (robotics, automotive, aerospace . . . )

e.g. passive m-s-d system mapping input \( F \) to output \( \dot{y} \):

\[
\begin{align*}
& m\ddot{x} + c(\dot{x}) + k(x) = F \\
& \text{sign}(k(y)) = \text{sign}(y) \\
& \text{sign}(c(\dot{y})) = \text{sign}(\dot{y})
\end{align*}
\]

\[
V = \frac{1}{2}m\dot{y}^2 + \int_0^y k(x) \, dx \quad \Rightarrow \quad \dot{V} = F\dot{y} - \dot{y}c(\dot{y})
\]
Passivity is useful for determining storage functions for feedback systems

**Closed-loop system** with passive subsystems $S_1, S_2$:

\[
\begin{align*}
S_1 &: \quad V_1 \geq 0 \quad \dot{V}_1 = y_1 u_1 - g_1 \\
S_2 &: \quad V_2 \geq 0 \quad \dot{V}_2 = y_2 u_2 - g_2
\end{align*}
\]

\[
\begin{align*}
V_1 + V_2 &\geq 0 \\
\dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\
&= y_1 (-y_2) + y_2 y_1 - g_1 - g_2 \\
&= -g_1 - g_2 \\
&\leq 0
\end{align*}
\]

\[\implies V = V_1 + V_2 \text{ is a Lyapunov function for the closed-loop system if } V \text{ is a p.d. function of the system state}\]
Interconnected passive systems

- **Parallel connection:**

  \[ V_1 + V_2 \geq 0 \]

  \[ \dot{V}_1 + \dot{V}_2 = y_1 u_1 + y_2 u_2 - g_1 - g_2 \]

  \[ = (y_1 + y_2)u - g_1 - g_2 \]

  \[ = yu - g_1 - g_2 \]

  \[ \Downarrow \]

  Overall system from \( u \) to \( y \) is passive

- **Feedback connection:**

  \[ V_1 + V_2 \geq 0 \]

  \[ \dot{V}_1 + \dot{V}_2 = y_1 u_1 + y_2 u_2 - g_1 - g_2 \]

  \[ = y(u - y_2) + y_2 y - g_1 - g_2 \]

  \[ = yu - g_1 - g_2 \]

  \[ \Downarrow \]

  Overall system from \( u \) to \( y \) is passive
Passive linear systems

Transfer function:

\[
\frac{Y(s)}{U(s)} = H(s)
\]

\[\begin{align*}
\text{u} & \rightarrow H(s) & \text{y} \\
\end{align*}\]

\[\uparrow\]

\[\text{frequency domain criterion for passivity}\]

\[\star\text{ H must be stable, otherwise } V(t) = \int_0^t yu \, dt \text{ is not defined for all } u\]

\[\star\text{ From Parseval’s theorem:}\]

\[\text{Re}[H(j\omega)] \geq 0 \iff \int_0^t yu \, dt \geq 0 \text{ for all } u(t) \text{ and } t\]

\[\star\text{ H is passive if and only if}\]

\[\begin{align*}
\text{(i). } & \text{Re}(p_i) \leq 0, \text{ where } \{p_i\} \text{ are the poles of } H(s) \\
\text{(ii). } & \text{Re}[H(j\omega)] \geq 0 \text{ for all } 0 \leq \omega \leq \infty
\end{align*}\]
Passive linear systems

Transfer function: \[ \frac{Y(s)}{U(s)} = H(s) \]

- \( H \) is dissipative if and only if \( \Re(p_i) \leq 0 \) and
  \[ \Re[H(j\omega)] > 0 \text{ for all } 0 \leq \omega < \infty \]

- Kalman-Yakubovich-Popov (KYP) Lemma:

  If \( H \) is dissipative, then there exists \( P > 0 \) such that
  \[ V = x^\top Px \text{ and } \dot{V} = yu - x^\top Qx, \quad Q > 0 \]

- \( x \) is the state (of any controllable state space realization) of \( H \)
- \( x = 0 \) is globally asymptotically stable with passive output feedback
Linear system + static nonlinearity

What are the conditions on $H$ and $\phi$ for closed-loop stability?

- A common problem in practice, due to e.g.
  - actuator saturation (valves, dc motors, etc.)
  - sensor nonlinearity

- Determine closed-loop stability given:
  - $\phi$ belongs to sector $[a, b]$  
    \[
a \leq \frac{\phi(y)}{y} \leq b
    \]
    “Absolute stability problem”

\[
Y(s) = H(s)U(s)
\]

$H$ linear:

$\phi$ static nonlinearity:

$z = \phi(y)$
Aizerman’s conjecture (1949):

Closed-loop system is stable if stable for $\phi(y) = ky$, $a \leq k \leq b$
false (necessary but not sufficient)

Sufficient conditions for closed-loop stability:

- Popov criterion (1960)
- Circle criterion

based on passivity

The passivity approach:

1. If $H$ is dissipative (i.e. if $\text{Re}[H(j\omega)] > 0$ and $H$ is stable), then:
$$
V = x^T Px \\
\dot{V} = yu - x^T Qx
$$
for some $P > 0$, $Q > 0$

$$
= -y\phi(y) - x^T Qx
$$

$x = \text{state of } H$

2. If $\phi$ belongs to sector $[0, \infty)$, then:
$$
y\phi(y) \geq 0
$$

$(1) \& (2) \implies \dot{V} \leq -x^T Qx$

$x = 0$ is globally asymptotically stable
Use loop transformations to generalize the approach for \[ \begin{cases} H \text{ not passive} \\ \phi \notin [0, \infty) \end{cases} \]

\[ e^{H - y} \leftrightarrow \text{equiv. to} \]

\[ a \]

\[ \phi \in [a, b] \quad a, b \text{ arbitrary} \]

\[ H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b - a} \]
Circle criterion

To make $H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b - a}$ dissipative, need:

(i). $H'$ stable $\iff$ $\frac{H(j\omega)}{1 + aH(j\omega)}$ stable

Nyquist plot of $H(j\omega)$ goes through $\nu$ anti-clockwise encirclements of $-1/a$ as $\omega$ goes from $-\infty$ to $\infty$

($\nu = \text{no. poles of } H(j\omega) \text{ in RHP}$)

(ii). $\text{Re}[H'(j\omega)] > 0 \iff \begin{cases} H(j\omega) \text{ lies outside } D(a, b) & \text{if } ab > 0 \\ H(j\omega) \text{ lies inside } D(a, b) & \text{if } ab < 0 \end{cases}$
Graphical interpretation of circle criterion

\( x = 0 \) is globally asymptotically stable if:

\[
\begin{align*}
\star & \quad 0 < a < b \\
& \quad H(j\omega) \text{ lies in shaded region and does } \nu \text{ anti-clockwise encirclements of } D(a, b) \\
\star & \quad b > a = 0 \\
& \quad H(j\omega) \text{ lies in shaded region and } \nu = 0 \text{ (can’t encircle } -1/a) \\
\star & \quad a < 0 < b \\
& \quad H(j\omega) \text{ lies in shaded region and } \nu = 0 \text{ (can’t encircle } -1/a) \\
\star & \quad a < b < 0 \\
& \quad -H(j\omega) \text{ lies in shaded region and does } \nu \text{ anti-clockwise encirclements of } D(-b, -a)
\end{align*}
\]
Circle criterion

- Circle criterion is equivalent to Nyquist criterion for $a = b > 0$
  \[ D(a, b) = -\frac{1}{a} \quad \text{(single point)} \]

- Circle criterion is only sufficient for closed-loop stability for general $a, b$

- Results apply to time-varying static nonlinearity: $\phi(y, t)$
Example: Active suspension system

Active suspension system for high-speed train:

\[
Q = \phi(u) \\
\dot{x}_a = Q/A
\]

\(u\) : valve input signal  
\(Q\) : flow rate  
\(\phi\) : valve characteristics, \(\phi \in [0.005, 0.1]\)  
\(A\) : actuator working area

Force exerted by suspension system on carriage body: \(F_{\text{susp}}\)

\[
F_{\text{susp}} = k(x_a - x) + c(\dot{x}_a - \dot{x}) \\
= \left( k \int Q \, dt + cQ \right)/A - kx - c\dot{x}, \quad Q = \phi(u)
\]

Design controller to compensate for the effects of (constant) unknown load on displacement \(x\) despite uncertain valve characteristics \(\phi(u)\).
Active suspension system contd.

Dynamics:

\[ F_{\text{susp}} - F = m\ddot{x} \]

\[ \implies m\ddot{x} + c\dot{x} + kx = \left( k \int_0^t Q\,dt + cQ \right)/A - F, \quad Q = \phi(u) \]

\( F \): unknown load on suspension unit

\( m \): effective carriage mass

Transfer function model:

\[ X(s) = \frac{cs + k}{ms^2 + cs + k} \cdot \frac{Q(s)}{As} - \frac{F}{ms^2 + cs + k} \quad Q = \phi(u) \]

Try linear compensator \( C(s) \):

\[ U(s) = C(s)E(s) \quad e = -x, \quad \text{setpoint: } x = 0 \]
For constant $F$, we need to stabilize the closed-loop system:

\[
\begin{align*}
\text{linear system:} & \quad H(s) = \frac{cs + k}{As(ms^2 + cs + k)} \cdot C(s) \\
\text{static nonlinearity:} & \quad \phi \in [0.005, 0.1] \\
\text{P+D compensator (no integral term needed):} & \quad C(s) = K(1 + \alpha s) \quad \implies \quad H(s) = \frac{K(1 + \alpha s)(cs + k)}{As(ms^2 + cs + k)} \\
\text{H open-loop stable (\(\nu = 0\))}
\end{align*}
\]

From the circle criterion, closed-loop (global asymptotic) stability is ensured if:

\[
H(j\omega) \text{ lies outside } D(0.005, 0.1)
\]

sufficient condition: $\text{Re}[H(j\omega)] > -10$
Nyquist plot of $H(j\omega)$ for $K = 1$ and $\alpha = 0, 0.2, 0.4, 0.8$:

To maximize gain margin:

choose $\alpha = 0.2$  ← allows for largest $K$

$K \leq \frac{10}{3.4} = 2.94$
Summary

At the end of the course you should be able to do the following:

- Understand the basic Lyapunov stability definitions (lecture 1)
- Analyse stability using the linearization method (lecture 2)
- Analyse stability by Lyapunov’s direct method (lecture 2)
- Determine convergence using Barbalat’s Lemma (lecture 3)
- Understand how invariant sets can determine regions of attraction (lecture 3)
- Construct Lyapunov functions for linear systems and passive systems (lecture 4)
- Use the circle criterion to design controllers for systems with static nonlinearities (lecture 4)