Equilibrium points

1. (a). Solving $\dot{x} = \sin^4 x - x^3 = 0$ for $x$ gives $x = 0$ as an equilibrium point. This is the only equilibrium because there is only one point $(x = 0)$ where $\sin x = x$ since

$$|\sin x| < |x| < 1 \implies |\sin x|^4 < |x|^3 \text{ for all } |x| \leq 1, \ x \neq 0$$

$$|\sin x| \leq 1 \implies |\sin x|^4 < |x|^3 \text{ for all } |x| > 1$$

(b). In terms of state variables $(x_1, x_2) = (x, \dot{x})$:

$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = \ddot{x} = -(x_1 - 1)^2 x_2^5 - x_1^2 + \sin(\pi x_1/2)$$

At an equilibrium point $\dot{x}_1 = \dot{x}_2 = 0$. But $\dot{x}_1 = 0$ implies $x_2 = 0$, so

$$\dot{x}_2 = 0 \implies x_1^2 - \sin(\pi x_1/2) = 0 \implies x_1 = 0 \text{ or } 1$$

Therefore equilibrium points are $(x_1, x_2) = (x, \dot{x}) = (0, 0)$ and $(1, 0)$. 
Lyapunov’s direct method, invariant sets and linearization

2. To explain the significance of constants $a, b, c$, we first give a derivation of the dynamics (this is not asked for in the question). The angular momentum of the craft in $xyz$-coordinates (Fig. 2) is given by

$$H = I \omega, \quad I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where $I_x, I_y, I_z$ are the moments of inertia about $x, y,$ and $z$-axes (assumed to be aligned with the spacecraft’s principle axes). Since there is no torque acting on the craft:

$$\frac{d}{dt}(I \omega) = I \dot{\omega} + \omega \times I \omega = 0$$

(where the $\omega \times I \omega$ term is needed because $xyz$-coordinates are fixed to and hence rotate with the spacecraft). So the full dynamics are given by

$$\dot{\omega}_x = a \omega_y \omega_z \quad \dot{\omega}_y = -b \omega_x \omega_z \quad \dot{\omega}_z = c \omega_x \omega_y$$

$$a = (I_y - I_z) / I_x, \quad b = (I_x - I_z) / I_y, \quad c = (I_x - I_y) / I_z$$

and the constants $a, b, c$ are all positive if $I_x > I_y > I_z$.

Figure 2: Rotating spacecraft.

(a). Equilibrium points: $\dot{\omega}_x = 0 \iff \dot{\omega}_y = 0$ or $\dot{\omega}_z = 0$, i.e. at least two of $\omega_x, \omega_y$ and $\omega_z$ must be zero for $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Therefore
every point in state space lying on the $\omega_x$-axis, the $\omega_y$-axis, or the $\omega_z$-axis is an equilibrium point.

(b). To show stability of the equilibrium at $\omega = 0$, try $V = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$ as a Lyapunov function. Clearly $V$ is positive definite if $p, q, r$ are all positive. Also

$$\dot{V} = 2(p\omega_x \dot{\omega}_x + q\omega_y \dot{\omega}_y + r\omega_z \dot{\omega}_z)$$

$$= 2(pa - qb + rc)\omega_x \omega_y \omega_z$$

Hence choosing $p, q, r$ so that

$$p > 0, \quad q > 0, \quad r > 0, \quad \text{and} \quad pa - qb + rc = 0,$$

(which is always possible since $q = (pa + rc)/b$ is positive for any chosen positive $p, r$), results in $\dot{V} = 0$, implying that $\omega = 0$ is a stable equilibrium point by Lyapunov’s direct method.

(c). Differentiating the function

$$V = c\omega_y^2 + b\omega_z^2 + [2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)]^2$$

(for constant $\omega_0$) with respect to $t$ along system trajectories yields

$$\dot{V} = 2c\omega_y \dot{\omega}_y \underbrace{+ 2b\omega_z \dot{\omega}_z}_{=0}$$

$$+ 2[2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)] \underbrace{(4ac\omega_y \dot{\omega}_y + 2ab\omega_z \dot{\omega}_z + 2bc\omega_x \dot{\omega}_x)}_{=0}$$

i.e. $\dot{V} = 0$. Also $V = 0$ only if $\omega = (\pm \omega_0, 0, 0)$, and $V > 0$ whenever $\omega_x \neq \pm \omega_0, \omega_y \neq 0$ or $\omega_z \neq 0$, so that $V$ is a locally positive definite function centered at the equilibrium $(\pm \omega_0, 0, 0)$. Therefore $\dot{V} = 0$ implies that every point on the $\omega_x$-axis in state space is a stable equilibrium, and hence that rotation at any constant velocity about the $x$-axis alone is stable.

[Note that rotational motion about the $z$-axis is likewise stable since $a, c$ and $\omega_x, \omega_z$ can be swapped in the dynamics and in the definition of $V$. However rotation about the $y$-axis is unstable, as shown by the
linearized system at $\omega = (0, \omega_0, 0)$:
\[
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix} \approx
\begin{bmatrix}
0 & 0 & a\omega_0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
\]
which has eigenvalues $\pm \sqrt{a c \omega_0}$ and 0, and is therefore unstable.

3. (a). The positive definite function $V = \frac{1}{2} x^2$ has derivative:
\[
\dot{V} = x \ddot{x} = -xb(x)
\]
which is negative definite due to $xb(x) > 0$ whenever $x \neq 0$. Therefore $x = 0$ is asymptotically stable, and since $V \to \infty$ as $x \to \infty$ it follows that $x = 0$ is globally asymptotically stable by Lyapunov's direct method.

(b). At an equilibrium point $\dot{x} = 0$. Hence $\ddot{x} = -c(x) = 0$ implies $x = 0$ since the condition $xc(x) > 0$ whenever $x \neq 0$ implies that $c(x)$ can only be equal to zero if $x = 0$. Therefore the only equilibrium point is the origin of state space: $(x, \dot{x}) = (0, 0)$.

The function $V(x, \dot{x})$ is positive definite and has derivative
\[
\dot{V} = \dot{x} \ddot{x} + c(x) \dot{x} = -\dot{x} b(\dot{x}) \leq 0
\]
and hence $(x, \dot{x}) = (0, 0)$ is stable by Lyapunov's direct method.

To apply the local invariant set theorem, we need to show that: (i) the level sets $\{(x, \dot{x}) : V(x, \dot{x}) \leq V_0\}$ are bounded for some $V_0$; (ii) $\dot{V} \leq 0$; (iii) the system dynamics are continuous and $V$ is continuously differentiable in $x$ and $\dot{x}$. Here (i) is satisfied because $V$ is increasing in both $x$ (since $\text{sign}(c(x)) = \text{sign}(x)$) and $\dot{x}$; (ii) is demonstrated above; and (iii) holds since $b(\dot{x})$, $c(x)$, $\partial V/\partial \dot{x} = \dot{x}$, and $\partial V/\partial x = c(x)$ are all continuous functions of $x$ and $\dot{x}$. Let $\mathcal{R} = \{(x, \dot{x}) : V = 0\}$ and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$, then
\[
\mathcal{R} = \{(x, \dot{x}) : \dot{x} = 0\}
\]
and since \( \ddot{x} = 0 \) is necessary in order that the state remains in \( \mathcal{R} \), we have

\[
\mathcal{M} = \mathcal{R} \cap \{(x, \dot{x}) : \ddot{x} = 0\} = \{(x, \dot{x}) : c(x) = 0\} = \{(0, 0)\}.
\]

From the local invariant set theorem, \((x, \dot{x})\) therefore converges asymptotically to \( \mathcal{R} \) from all initial conditions within any bounded level set of \( V \), implying that \((0, 0)\) is asymptotically stable.

To show global asymptotic stability we need \( V \) to be radially unbounded (in order to apply the global invariant set theorem) or equivalently the level sets of \( V \) must cover the entire state space as \( V_0 \to \infty \). This condition requires

\[
\int^x c(s) \, ds \to \infty \text{ as } x \to \infty.
\]

4. (a). The equilibrium points can be found by solving \( \dot{x}_1 = \dot{x}_2 = 0 \) for \( x_1 \) and \( x_2 \):

\[
\dot{x}_1 = 0 \quad \implies \quad x_2 = 0 \\
\dot{x}_1 = \dot{x}_2 = 0 \quad \implies \quad x_1(x_1^2 - 1) = 0 \quad \implies \quad x_1 = 0, 1, -1.
\]

Hence the equilibrium points are \((x_1, x_2) = \{(0, 0), (1, 0), (-1, 0)\}\).

(b). The system and function \( V \) have the following properties.

(i). \( V, \dot{x}_1 \) and \( \dot{x}_2 \) are continuous functions of \( x_1 \) and \( x_2 \).

(ii). The level sets: \( \{(x_1, x_2) : V \leq V_0\} \) are finite and \( V \) is radially unbounded since \( V \to \infty \) as \( |x_1| \to \infty \) and/or \( |x_2| \to \infty \).

(iii). Along system trajectories, \( V \) has derivative

\[
\dot{V}(x_1, x_2) = x_2 \dot{x}_2 + x_1(x_1^2 - 1) \dot{x}_1 \\
= -x_2^2(x_1 - 1)^2 - x_1x_2(x_1^2 - 1) + x_1x_2(x_1^2 - 1) \\
= -x_2^2(x_1 - 1)^2 \\
\leq 0.
\]
Using the global invariant set theorem, (i)-(iii) imply that every state trajectory tends to an invariant set on which \( \dot{V} = 0 \). (The same conclusion can be reached using the local invariant set theorem, since the level sets of \( V \) can be made arbitrarily large by choosing \( V_0 \) sufficiently large.)

From (iii), \( \dot{V}(x_1, x_2) = 0 \) is satisfied on the lines \( x_2 = 0 \) and \( x_1 = 1 \). The invariant sets within these lines are defined by \( \dot{x}_2 = 0 \) (on \( x_2 = 0 \)) and \( \dot{x}_1 = 0 \) (on \( x_1 = 1 \)). But

\[
\begin{align*}
\{ x_2 = 0 \} \cap \{ \dot{x}_2 = 0 \} &\Rightarrow x_1 = 0, 1, -1, \\
\{ x_1 = 1 \} \cap \{ \dot{x}_1 = 0 \} &\Rightarrow x_2 = 0
\end{align*}
\]

and every state trajectory therefore tends asymptotically to one of the three equilibrium points identified in (a).

(c). Writing the system dynamics in the form \( \dot{x} = f(x) \), \( x = [x_1 \ x_2]^T \)

where the Jacobian matrix of \( f \) is

\[
\frac{\partial f}{\partial x}(x) = \begin{bmatrix}
0 & 1 \\
-2x_2(x_1 - 1) - (3x_1^2 - 1) & -(x_1 - 1)^2
\end{bmatrix},
\]

the linearization of the system at \( x_1 = x_2 = 0 \) is given by

\[
\dot{x} = Ax, \quad A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.
\]

\( A \) has eigenvalues \(-1/2 \pm \sqrt{3}/2\), and it follows that the origin is an unstable equilibrium of the nonlinear system, by Lyapunov’s linearization method.

(d). \( V \) has local minimum points at \((x_1, x_2) = (-1, 0)\) and \((1, 0)\) (since

\[
\nabla V = \begin{bmatrix} x_1^3 - x_1 \\ x_2 \end{bmatrix} = 0 \quad \frac{\partial^2 V}{\partial x^2} = \begin{bmatrix} 3x_1^2 - 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} > 0
\]

at \((x_1, x_2) = (-1, 0)\) and \((1, 0)\)). Hence \( V + \frac{1}{4} \) is locally positive definite at \((x_1, x_2) = (-1, 0)\) and \((1, 0)\), and from Lyapunov’s direct method these equilibrium points are therefore stable because \( \dot{V} \leq 0 \).
Other approaches for (d): The equilibrium at \((-1, 0)\) can be shown to be stable using the linearization method, since the linearization at this point is stable. However the linearization about \((1, 0)\) has eigenvalues \(\pm i\sqrt{2}\), and therefore does not allow any conclusions to be made about the stability of this equilibrium for the nonlinear system.

5. (a). Using matrices \(A, B, K\) and the given matrix \(P\) we get (2 marks):

\[
Q = -(A - BK)^TP - P(A - BK) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\]

where

\[
\text{eig}(P) = \lambda : \lambda^2 - 3\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{3}{2} \pm \frac{\sqrt{5}}{2}
\]

\[
\text{eig}(Q) = \lambda : \lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \quad \lambda = 1, 3
\]

The equilibrium \(x = 0\) is locally asymptotically stable since:

- the linearized closed loop system about \(x = 0\) is \(\dot{x} = (A - BK)x\)
- \((A - BK)^TP + P(A - BK) = -Q\) for positive definite \(P, Q\) implies \(\dot{x} = (A - BK)x\) is stable, i.e. \(\text{Re}[\text{eig}(A - BK)] < 0\)
- so the nonlinear closed loop system is locally a.s.

(b). From \(V = x^TPx\) and \(\dot{x} = (A - BK)x - x(Kx)\) we get

\[
\dot{V} = x^T[(A - BK)^TP + P(A - BK)]x - (Kx)x^T(P + P)x
\]

\[
\leq -x^TQx - 2|Kx|^2x^TPx
\]

But \(x^TPx - x^TQx = x^T(P - Q)x = -x_2^2 \leq 0\),

so \(\dot{V} \leq -x^TQx + 2|Kx|^2x^TPx\).

(c). \(\dot{V} \leq -x^TQx(1 - 2|Kx|)\), so \(\dot{V}\) is negative definite in the region where \(|Kx| < \frac{1}{2}\), which is the strip between the dashed lines in the figure below.
Any level set of $V$ contained entirely within this strip is invariant and hence is a region of attraction for $x = 0$.

The level sets $\Omega$ are ellipsoidal, centred on the origin, and decrease in size as $\alpha$ is reduced. Hence $\Omega$ must be invariant for small enough $\alpha$.

**Linear and passive systems**

6. Let $\Phi = A + \mu I$, then $A^T P + PA + 2\mu P = -Q$ implies

$$
\Phi^T P + P\Phi = A^T P + PA + 2\mu P = -Q,
$$

so $P, Q > 0$ imply that $\text{Re}\{\text{eig}(\Phi)\} < 0$, so that $\text{Re}\{\text{eig}(A + \mu I)\} < 0$, and therefore $\text{Re}\{\text{eig}(A)\} < -\mu$

(since $A = V\Lambda V^{-1} \implies \Phi = V(\Lambda - \mu I)V^{-1}$).

7. (a). Differentiating $V_1$ with respect to $t$ gives:

$$
\dot{V}_1 = \frac{x_2 e}{L(x_2)} - \frac{R_1}{L^2(x_2)} x_2^2 = \dot{x}_1 e - \frac{R_1}{L^2(x_2)} x_2^2
$$

and since $V \geq 0$, this implies that the dynamic system with $e$ as input and $\dot{x}_1$ as output is passive (in fact it is dissipative).

(b). Let $x_3$ and $x_4$ be respectively the charge on the capacitor and flux in the inductor in the right-hand branch of the circuit, and define

$$
V_2(x_3, x_4) = \int_0^{x_4} \frac{x}{L(x)} dx + \int_0^{x_3} \frac{x}{C(x)} dx.
$$

Differentiating w.r.t. $t$ gives $\dot{V}_2 = \dot{x}_3 e - R_2 x_4^2 / L^2(x_4)$. Therefore, defining $V = V_1 + V_2$ and using the fact that $\dot{x}_1 + \dot{x}_3 = i$ (since the
currents in the two branches of the circuit must sum to \( i \), we get

\[
V = \int_0^{x_2} \frac{x}{L(x)} \, dx + \int_0^{x_4} \frac{x}{L(x)} \, dx + \int_0^{x_1} \frac{x}{C(x)} \, dx + \int_0^{x_3} \frac{x}{C(x)} \, dx
\]

\[
\dot{V} = ie - \frac{R_1}{L^2(x_2)} x_2^2 - \frac{R_2}{L^2(x_4)} x_4^2.
\]

and \( V \geq 0 \) since \( V_1, V_2 \geq 0 \).

Opening the switch forces \( i = 0 \), so

\[
\dot{V} = -\frac{R_1}{L^2(x_2)} x_2^2 - \frac{R_2}{L^2(x_4)} x_4^2
\]

and since the level sets \( \{(x_1, x_2, x_3, x_4) : V \leq \bar{V}\} \) are bounded (when \( \bar{V} \) is sufficiently small), it follows from the local invariant set theorem that the system is (locally) asymptotically stable.

Specifically, \( x = (x_1, x_2, x_3, x_4) \) must converge to the largest invariant set within the set of states such that \( \dot{V} = 0 \), i.e. \( x_2 = x_4 = 0 \) and \( \dot{x}_2 = \dot{x}_4 = 0 \), implying that \( x \) converges asymptotically to a steady state such that \( x_1/C(x_1) = x_3/C(x_3) = 0 \) and \( x_2, x_4 = 0 \). This asymptotic stability property is global if \( V_1, V_2 \) are radially unbounded. Note also that the same analysis can be applied to any number of LCR branches connected in parallel.

8. (a). The rectangular region containing \( G(j\omega) \) lies within \( D(a, b) \) if \( a = -\frac{1}{3} \) and \( b = \frac{1}{2} \), since \( D(a, b) \) is then just touching its corners (Fig. 3).

The open-loop system is stable, and the circle criterion therefore implies that the closed-loop system with \( u = -\phi(y) \) will be asymptotically stable if \( \phi \) lies in the sector \([0, 1/2] \).

Clearly this is not the only sector bound for \( \phi \) for which the closed-loop system is guaranteed to be stable by the circle criterion. In fact a family of discs \( D(a, b) \) containing \( G(j\omega) \) is generated as \( a \) is increased from \(-1/3\), and to allow for the largest possible value of \( b \) we need to set \( a = 0 \) and \( b = -1 \), corresponding to sector bounds \( \phi \in [0, 1] \).
(b). Closed-loop stability does not apply to nonlinearities $\phi$ bounded by the union of the two sectors defined in part (a), i.e. $[-\frac{1}{3}, 1]$, since this includes nonlinearities not belonging to either of the sectors $[-\frac{1}{3}, \frac{1}{2}]$ and $[0, 1]$. In particular, the disc centred on the real axis and intersecting the real axis at $-1$ and $3$ does not entirely contain the box in which $G(j\omega)$ is known to lie, so it cannot be concluded from the circle criterion that the closed loop system will be stable.

Figure 3: Bounds on the Nyquist plot of $G(j\omega)$. 