C24 DS1 Dynamical Systems 1

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You are expected to refer to textbooks as well as the lecture notes when attempting the questions on this sheet. Some questions suggest using numerical methods in MATLAB or Mathematica.

Questions

1. Equilibria of time-varying systems. Consider the system

$$\dot{x} = -x + t$$

(a) Show that if time is frozen, $x = t$ is an 'instantaneous' equilibrium of this system.

(b) Show that the solution to this differential equation is

$$x(t) = t - 1 + e^{-t}(x_0 + 1)$$

where $x(0) = x_0$.

(c) What is the equilibrium of the nonautonomous system? Draw the trajectories of the system.

(This question shows that looking at 'instantaneous' equilibria can result in incorrect conclusions.)

2. Nature of Equilibria of Maps Find the stable, unstable and centre subspaces of the system

$$\begin{aligned} x_{k+1} &= \lambda x_k + y_k \\ y_{k+1} &= \mu y_k \end{aligned}$$

where $\lambda \neq 0$ and $\mu \neq 0$, for the following cases:

(a) $|\lambda|, |\mu| > 1$

(b) $|\lambda|, |\mu| < 1$

(c) $|\lambda| > 1, |\mu| < 1$

(d) $|\lambda| = 1, |\mu| > 1.$

3. Linear Flows

(a) For the following systems, solve the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, determine the stable, unstable and centre subspaces and sketch the phase portraits:

(i) $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$

(ii) $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$

(iii) $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x}$

(iv) $\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$

(b) Show that, for a real scalar θ,

$$\exp\left(\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(c) Consider a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ has eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$. Representing the initial condition by

$$\mathbf{x}(0) = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$$

show that

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n.$$

(d) For a 2-D linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, show that the eigenvalues of $\mathbf{A}$ satisfy

$$\lambda^2 - \tau\lambda + D = 0$$

where τ is the Trace of $\mathbf{A}$ and D is its determinant. Then, show on a graph with axes D and τ, the regions where one expects to have spirals (stable or unstable), nodes (stable or unstable), saddles and centres.

4. Nonlinear Flows

(a) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1(3 - x_1 - x_2) \\ \dot{x}_2 &= x_2(x_1 - 1) \end{aligned}$$

Draw the phase plane of this system, showing clearly the position of the equilibria, the behaviour of the system close to them and the far-from-equilibrium behaviour.

(b) Draw the phase plane of the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_1 x_2 \\ \dot{x}_2 &= 0.5 x_2^2 + x_1 x_2 \end{aligned}$$

[Hint: plot the vector field using MATLAB or Mathematica.]

(c) Draw the phase plane of the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 \\ \dot{x}_2 &= x_2 \end{aligned}$$

[Hint: plot the vector field using MATLAB or Mathematica. Think about the trajectories of the system.]

(d) Draw the phase plane of the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 \end{aligned}$$

[Hint: plot the vector field using MATLAB or Mathematica. Think about the trajectories of the system.]

5. Polar transformations

(a) Show that a two-dimensional nonlinear system with states x_1 and x_2 can be written in polar coordinates using

$$\begin{aligned}\dot{r} &= \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}\\ \dot{\theta} &= \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}\end{aligned}$$

(b) Consider the following system

$$\begin{aligned}\dot{x}_1 &= -x_2 + ax_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= x_1 + ax_2(x_1^2 + x_2^2)\end{aligned}$$

What is the nature of the zero equilibrium as a function of a?

6. Nonlinear Centres

(a) Show that a system of the form $\ddot{x} = f(x)$ can be written as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= f(x_1)\end{aligned}$$

by appropriate choice of x_1, x_2.

(b) By considering the expression

$$x_1\dot{x}_1 + x_2\dot{x}_2$$

show that if the system is conservative, one can obtain a non-constant function $V(x_1, x_2)$ such that $\frac{d}{dt}V(x_1, x_2) = 0$ by integrating

$$-f(x_1)\dot{x}_1 + x_2\dot{x}_2 = 0.$$

(c) Apply your results to the system

$$\begin{aligned}\dot{z}_1 &= -z_2 - z_2^3\\ \dot{z}_2 &= z_1\end{aligned}$$

to obtain the function $V(z_1, z_2)$.

7. Lyapunov functions

(a) Show that the origin of the system

$$\begin{aligned} \dot{x} &= -y - x(x^2 + y^2) \\ \dot{y} &= x - y(x^2 + y^2) \end{aligned}$$

is globally asymptotically stable using Lyapunov's direct method.

(b) Using the function $V(x, y) = 2y^2 - 2x^2 + x^4$ show that the $(\pm 1, 0)$ equilibria of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 - \gamma y, \quad \gamma > 0 \end{aligned}$$

are stable. [In later lectures, we will show that these equilibria are actually asymptotically stable.]

(c) Show that the zero equilibrium of the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2^3 - 2x_2^4 \\ \dot{x}_2 &= -x_1 - x_2 + x_1 x_2 \end{aligned}$$

is globally asymptotically stable. [Hint: try a function of the form $V(x_1, x_2) = x_1^{\alpha_1} + k x_2^{\alpha_2}$.]

8. Hamiltonian and Gradient Systems

(a) Show that the system

$$\begin{aligned} \dot{x} &= 2\cos x + \cos y \\ \dot{y} &= 2\cos y + \cos x \end{aligned}$$

has a symmetry around the origin but that it is not conservative.

(b) Show that the system

$$\begin{aligned} \dot{x}_1 &= \sin x_2 \\ \dot{x}_2 &= x_1 \cos x_2 \end{aligned}$$

is a gradient system and find the system potential. Hence construct a system which is Hamiltonian, that is related to this gradient system.

Some Answers and Hints

2(a). $E^U = \mathbb{R}^2$, $E^S = \emptyset$, $E^C = \emptyset$.

3(b). Hint: Expand the exponential in series.

4(a). $(0, 0)$ is a saddle point
$(1, 2)$ is a counterclockwise stable spiral
$(3, 0)$ is a saddle point.

5. Hint: Differentiate $x_1^2 + x_2^2 = r^2$. $a = 0$: nonlinear centre.

6(c). $V(z_1, z_2) = \frac{1}{2}z_2^2 + \frac{1}{2}z_1^2 + \frac{1}{4}z_2^4$.

7(c). $V(x_1, x_2) = x_1^2 + x_2^4$.

8(a). The $(\pi/2, \pi/2)$ equilibrium is attracting.