C24.DS Examples 1

① Consider $\frac{dx}{dt} = -x + t \Rightarrow \frac{dx}{dt} + x = t$

ⓐ If time is frozen and $x(t) = t$ then $\frac{dx}{dt} + t = t \Rightarrow \frac{dx}{dt} = 0$, apparently. But this is impossible since $\frac{dx}{dt} = \frac{dt}{dt} = 1$.
We could consider this as a second state; let $x_1 = x$, $x_2 = t$. Then
$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = 1 \end{cases}$$
$\Rightarrow$ No equilibrium for any finite t

ⓑ Solve as you please, with integrating factors, etc.
We'll use Laplace transforms.

$$L\left\{\frac{dx}{dt} + x\right\} = L\{t\} \Rightarrow s\bar{x} - x_0 + \bar{x} = \frac{1}{s^2}$$

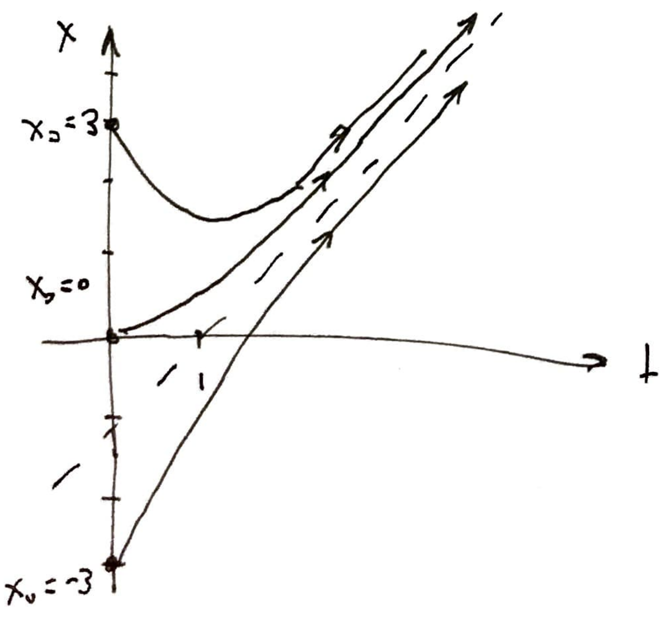
$$\Rightarrow \bar{x} = \frac{x_0}{1+s} + \frac{1}{s^2(1+s)}$$

$$x(t) = L^{-1}\left\{\frac{x_0}{1+s}\right\} + L^{-1}\left\{\frac{1}{s^2(1+s)}\right\} = x_0 e^{-t} + \int_0^t \int_0^t e^{-u}\,du\,dv$$

$$= x_0 e^{-t} + \int_0^t (1 - e^{-v})\,dv = x_0 e^{-t} + t - (1 - e^{-t})$$

$$\boxed{x(t) = t - 1 + e^{-t}(x_0 + 1)}$$

ⓒ There is not an 'equilibrium' per se but at long times we have asymptotic behaviour: $\boxed{\lim_{t \gg 1} x(t) = t - 1}$



C24. DS Examples 1

(2) Given discrete map $\begin{aligned} x_{k+1} &= \lambda x_k + y_k \\ y_{k+1} &= \mu y_k \end{aligned} \Rightarrow \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$

Let $\underline{\underline{A}} = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix}$ so that $\underline{x}_{k+1} = \underline{\underline{A}}\, \underline{x}_k$

Now observe $\underline{x}_1 = \underline{\underline{A}}\,\underline{x}_0$, $\underline{x}_2 = \underline{\underline{A}}\,\underline{x}_1 = \underline{\underline{A}}^2 \underline{x}_0, \ldots, \underline{x}_n = \underline{\underline{A}}^n \underline{x}_0$

So the task comes down to computing $\underline{\underline{A}}^n$.

Eigenvalues of $\underline{\underline{A}}$ are $\lambda_1 = \lambda$, $\lambda_2 = \mu$

For $\mu \neq \lambda$ there are two eigenvectors:

$$\underline{x}^\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{x}^\mu = \begin{bmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{bmatrix}$$

For $\mu = \lambda$ the matrix $\underline{\underline{A}}$ is degenerate so just one eigenvector, $\underline{x}^\lambda$

Assuming $\underline{\underline{A}}$ diagonalizable (i.e. non-degenerate),

$$\underline{\underline{A}} = \begin{bmatrix} 1 & \frac{1}{\mu-\lambda} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\lambda-\mu} \\ 0 & 1 \end{bmatrix} = \underline{\underline{V}}\, \underline{\underline{D}}\, \underline{\underline{V}}^{-1}$$

And $\underline{\underline{A}}^n = \underline{\underline{V}}\, \underline{\underline{D}}^n \underline{\underline{V}}^{-1}$, where $\underline{\underline{D}}^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix}$.

(a) $|\lambda|, |\mu| > 1$: Unstable equilibrium: $E^s = E^c = \emptyset$ (empty set)

$E^u = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{bmatrix} \right\}$ if $\mu \neq \lambda$, else just $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

(b) $|\lambda|, |\mu| < 1$: Asymptotically stable eq: $E^c = E^u = \emptyset$

$E^s = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{bmatrix} \right\}$ if $\mu \neq \lambda$, else just $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

(c) $|\lambda| > 1$, $|\mu| < 1$ (implies $\mu \neq \lambda$) saddle point: $E^c = \emptyset$

$E^s = \text{span}\left\{ \begin{bmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{bmatrix} \right\}$; $E^u = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

(d) $|\lambda| = 1$, $|\mu| > 1$ (implies $\mu \neq \lambda$): not stable, so $E^s = \emptyset$

$E^u = \text{span}\left\{ \begin{bmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{bmatrix} \right\}$; $E^c = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Note $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not necessarily isolated in this case; could be in E^c or E^u.

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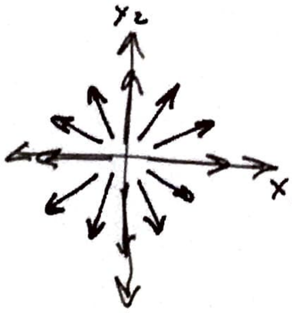
③ ⓘ $\dot{\underline{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} \Rightarrow \boxed{\underline{x}(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \underline{x}_0}$

$E^s = E^c = \phi \; ; \; E^u = \mathbb{R}^2$



ⓘⓘ $\dot{\underline{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underline{x} \Rightarrow$ $x_2(t) = x_2(0)e^t$

$\frac{dx_1}{dt} = x_1 + x_2 = x_1 + x_2(0)e^t$

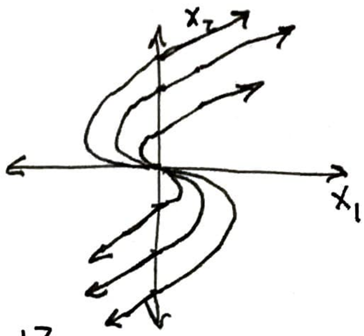
Laplace trans.

$\Rightarrow s\bar{x}_1 - x_1(0) - \bar{x}_1 = \frac{x_2(0)}{s-1} \Rightarrow \bar{x}_1 = \frac{x_1(0)}{s-1} + \frac{x_2(0)}{(s-1)^2}$

shift theorem

$\Rightarrow x_1(t) = x_1(0)e^t + e^t L^{-1}\left\{\frac{x_2(0)}{s^2}\right\} = x_1(0)e^t + x_2(0)te^t$

Thus $\boxed{\underline{x}(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \underline{x}_0}$ $E^s = E^c = \phi$; $E^u = \mathbb{R}^2$



Note for plot that $\dot{\underline{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ @ $\underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ⓘⓘⓘ $\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \underline{x}$ Eigenvalues are $\lambda \in \{2, 1, -1\}$ (Clear since matrix is lower triangular).

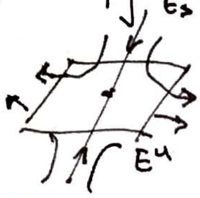
Eigenvectors: $\underline{x}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\underline{x}^1 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ $\underline{x}^{-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\begin{array}{l} E^c = \phi \\ E^s = \text{span}\{\underline{x}^{-1}\} \\ E^u = \text{span}\{\underline{x}^2, \underline{x}^1\} \end{array}}$

Put a 1 in first entry of $\underline{x}^1$ by multiplying by $\frac{3}{2}$. Then

$\underline{\underline{V}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$ and $\underline{\underline{V}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$; $\underline{\underline{A}} = \underline{\underline{V}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \underline{\underline{V}}^{-1}$

$\underline{x}(t) = e^{\underline{\underline{A}}t} \underline{x}_0 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \underline{x}_0$



$\underline{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ e^{2t} & e^{2t} & 0 \\ -\frac{1}{2}e^{-t} & 0 & e^{-t} \end{bmatrix} \underline{x}_0 \Rightarrow \boxed{\underline{x} = \begin{bmatrix} e^t & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 \\ \frac{1}{2}(e^t - e^{-t}) & 0 & e^{-t} \end{bmatrix} \underline{x}_0}$

C24. DS Examples 1.

(3) (a) (iv) $\dot{\underline{x}} = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\underline{x}$ matrix is symmetric; eigenvectors $\perp$

Eigenvalues: $(-1-\lambda)^2-1=0 \Rightarrow \lambda(\lambda+2)=0$

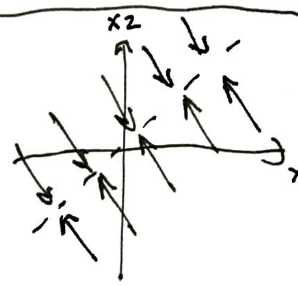
Eigenvector for $\lambda=0$: $\underline{x}^0 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$; $\underline{x}^{-2}$ is $\perp$: $\underline{x}^{-2} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$

$$\dot{\underline{x}} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\underline{x}$$

$$\underline{x} = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^{-2t} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\underline{x}_0 = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^{-2t} & -e^{-2t}\\ 1 & 1\end{bmatrix}\underline{x}_0$$

$$\underline{x}(t) = \frac{1}{2}\begin{bmatrix}1+e^{-2t} & 1-e^{-2t}\\ 1-e^{-2t} & 1+e^{-2t}\end{bmatrix}\underline{x}_0 \Rightarrow \boxed{\underline{x}(t) = e^{-t}\begin{bmatrix}\cosh(t) & \sinh(t)\\ \sinh(t) & \cosh(t)\end{bmatrix}}$$

$E^u = \emptyset$, $E^s = \operatorname{span}\left\{\begin{bmatrix}1\\ -1\end{bmatrix}\right\}$, $E^c = \operatorname{span}\left\{\begin{bmatrix}1\\ 1\end{bmatrix}\right\}$.



(b) $e^{\theta\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}} \Rightarrow \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = -i\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$, Hermitian.

for $\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$ eigenvalues are such that $\lambda^2 - (-i)i = 0$; $\lambda \in \{1, -1\}$

eigenvectors $\underline{x}^1 = \begin{bmatrix}i\\ 1\end{bmatrix}$ $\underline{x}^{-1} = \begin{bmatrix}-i\\ 1\end{bmatrix}$

So $$\exp\left\{\theta\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right\} = \exp\left\{-i\theta\begin{bmatrix}i/\sqrt{2} & -i/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-i/\sqrt{2} & 1/\sqrt{2}\\ i/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\right\}$$

$$= \begin{bmatrix}i/\sqrt{2} & -i/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}e^{-i\theta} & 0\\ 0 & e^{i\theta}\end{bmatrix}\begin{bmatrix}-i/\sqrt{2} & 1/\sqrt{2}\\ i/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}i & -i\\ 1 & 1\end{bmatrix}\begin{bmatrix}-ie^{-i\theta} & e^{-i\theta}\\ ie^{i\theta} & e^{i\theta}\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}e^{-i\theta}+e^{i\theta} & ie^{-i\theta}-ie^{i\theta}\\ -ie^{-i\theta}+ie^{i\theta} & e^{-i\theta}+e^{i\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \checkmark$$

(c) Clear, since $e^{\underline{\underline{A}}t}$ has same eigenvectors as $\underline{\underline{A}}$.

C24. VS. Examples 1

③ ⓐ If $\underline{\underline{A}}$ is 2×2 $\underline{\underline{A}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

Eigenvalues of $\underline{\underline{A}}$ are such that $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\det\left(\begin{bmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{bmatrix}\right) = (A_{11}-\lambda)(A_{22}-\lambda) - A_{12}A_{21}$$

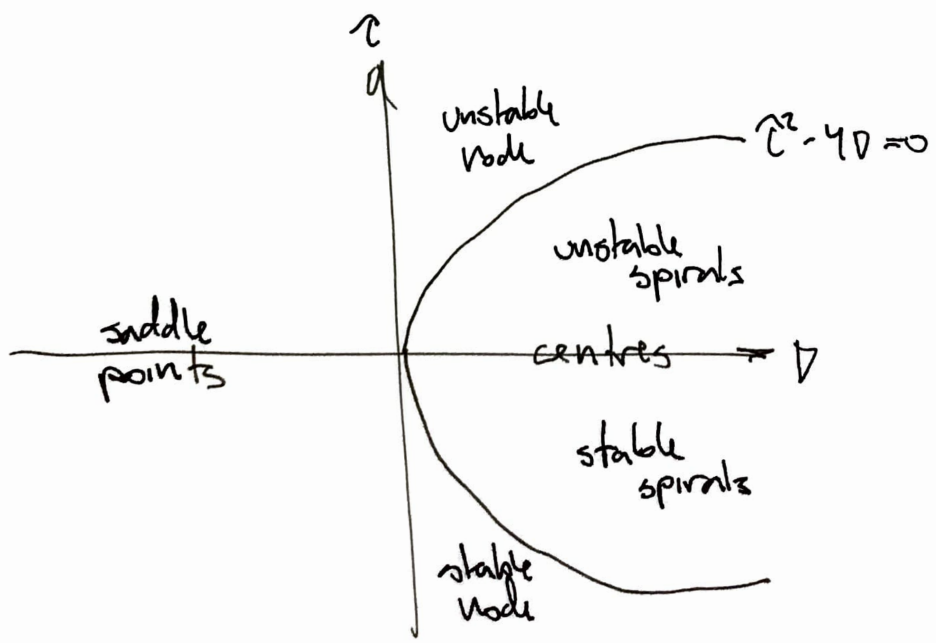
$$= \lambda^2 - (A_{11}+A_{22})\lambda + A_{11}A_{22} - A_{12}A_{21}$$

And since $tr(\underline{\underline{A}}) = \tau = A_{11} + A_{22}$

$\det(\underline{\underline{A}}) = \Delta = A_{11}A_{22} - A_{12}A_{21}$

$$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0 \Rightarrow \boxed{\lambda^2 - \tau\lambda + \Delta = 0}$$

Roots of this are $\lambda_{\pm} = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$



C24. DS Examples 1

(4) (a) $\dot{x}_1 = x_1(3 - x_1 - x_2)$
$\dot{x}_2 = x_2(x_1 - 1)$ has equilibria @ $\underline{x}^* \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Jacobian is $D\underline{f} = \begin{bmatrix} -2x_1 - x_2 + 3 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$

Thus $D\underline{f}|_{(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ <u>saddle</u> since $\lambda \in \{3, -1\}$

$D\underline{f}|_{(3,0)} = \begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix}$ <u>saddle</u> since $\lambda \in \{-3, 2\}$

$D\underline{f}|_{(1,2)} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$ $\det(D\underline{f}|_{(1,2)}) = -(1+\lambda)\lambda + 2 = 0$

$\lambda^2 - \lambda + 2 = 0 \quad \lambda \in -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$

<u>stable counterclockwise spiral</u>

(b) → (d) See Mathematica notebook

Examples Sheet 1: Solutions for Question 4

In[∘]:=
```
flow4a[x1_, x2_] := {x1 * (3 - x1 - x2), x2 * (x1 - 1)}
```

In[∘]:=
```
Jaco[x1_, x2_] :=
 {{D[flow4a[x, x2][[1]], x] /. x → x1, D[flow4a[x1, y][[1]], y] /. y → x2},
  {D[flow4a[x, x2][[2]], x] /. x → x1, D[flow4a[x1, y][[2]], y] /. y → x2}}
```

In[∘]:=
```
MatrixForm[Jaco[x1, x2]]
```

Out[∘]//MatrixForm=

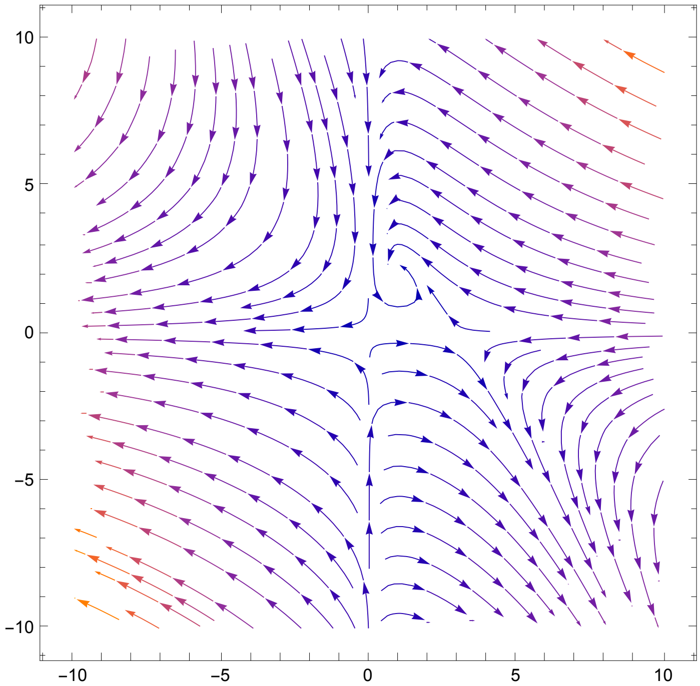
$$\begin{pmatrix} 3 - 2\,x1 - x2 & -x1 \\ x2 & -1 + x1 \end{pmatrix}$$

In[∘]:=
```
Eigenvectors[Jaco[3, 0]]
```

Out[∘]= {{1, 0}, {-3, 5}}

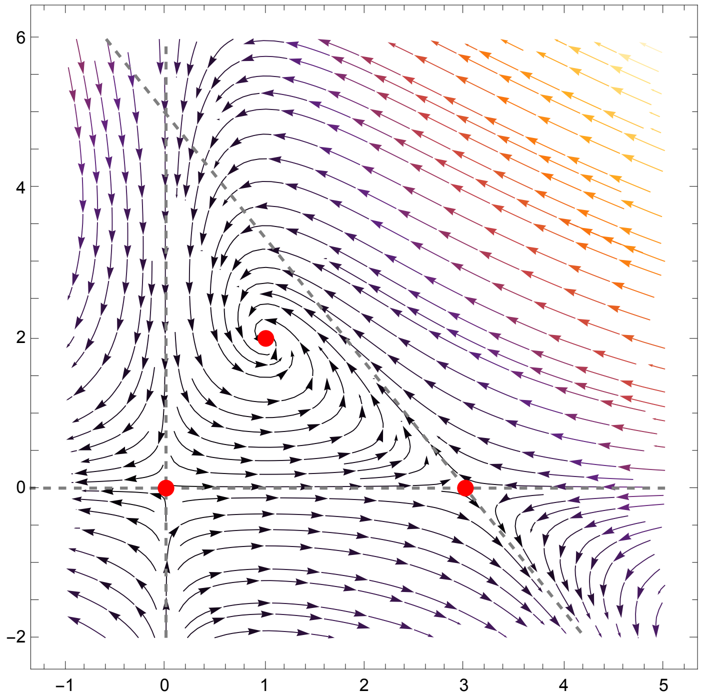
In[∘]:=
```
StreamPlot[flow4a[x1, x2], {x1, -10, 10}, {x2, -10, 10}]
```

Out[∘]=



In[•]:=
```
Show[StreamPlot[flow4a[x1, x2], {x1, -1, 5}, {x2, -2, 6}, PlotTheme → "Detailed",
   StreamColorFunction → "SunsetColors"], Plot[{0, 10^6 * x, -5 / 3 * (x - 3)},
   {x, -2, 5}, PlotRange → {{-1, 5}, {-2, 6}}, Frame → True, AspectRatio → 8 / 6,
   PlotStyle → {{Gray, Dashed}, {Gray, Dashed}, {Gray, Dashed}}],
  ListPlot[{Style[{0, 0}, {Red, PointSize → 0.025}], Style[{3, 0},
     {Red, PointSize → 0.025}], Style[{1, 2}, {Red, PointSize → 0.025}]}]]
```

Out[•]=



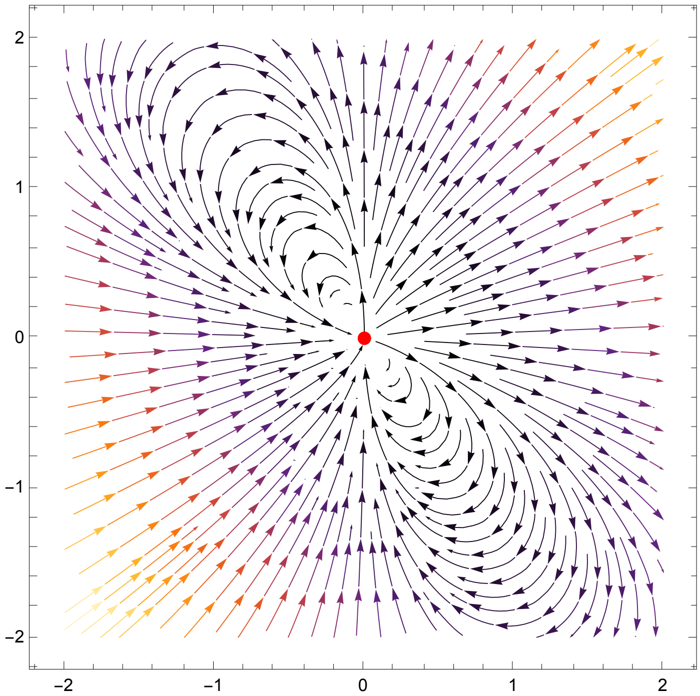
In[•]:=
```
flow4b[x1_, x2_] := {x1^2 + x1 * x2, x2^2 / 2 + x1 * x2}
```

In[•]:=
```
Solve[{flow4b[x1, x2][[1]] == 0, flow4b[x1, x2][[2]] == 0}, {x1, x2}]
```

Out[•]=
```
{{x1 → 0, x2 → 0}, {x1 → 0, x2 → 0}}
```

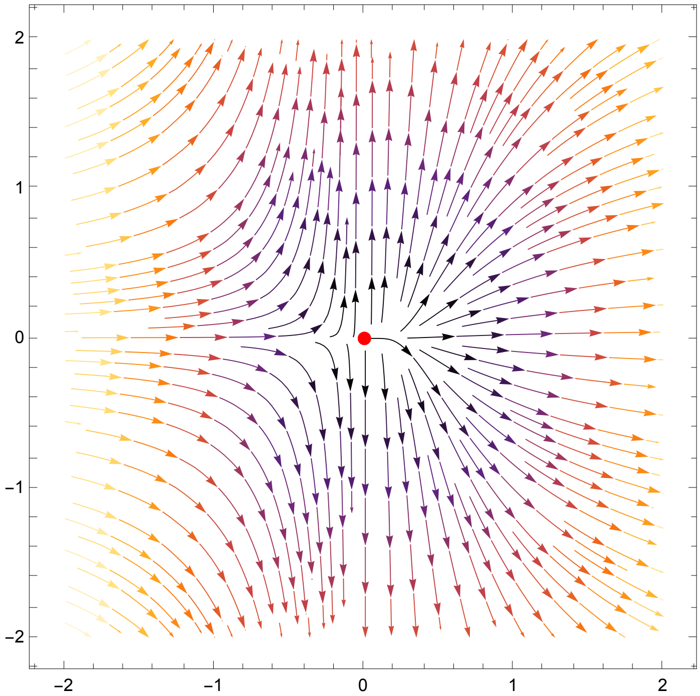
In[◦]:=
```
Show[StreamPlot[flow4b[x1, x2], {x1, -2, 2}, {x2, -2, 2},
   PlotTheme → "Detailed", StreamColorFunction → "SunsetColors"],
  ListPlot[{Style[{0, 0}, {Red, PointSize → 0.02}]}]]
```

Out[◦]=



In[◦]:=
```
flow4c[x1_, x2_] := {x1^2, x2}
Show[StreamPlot[flow4c[x1, x2], {x1, -2, 2}, {x2, -2, 2},
   PlotTheme → "Detailed", StreamColorFunction → "SunsetColors"],
  ListPlot[{Style[{0, 0}, {Red, PointSize → 0.02}]}]]
```

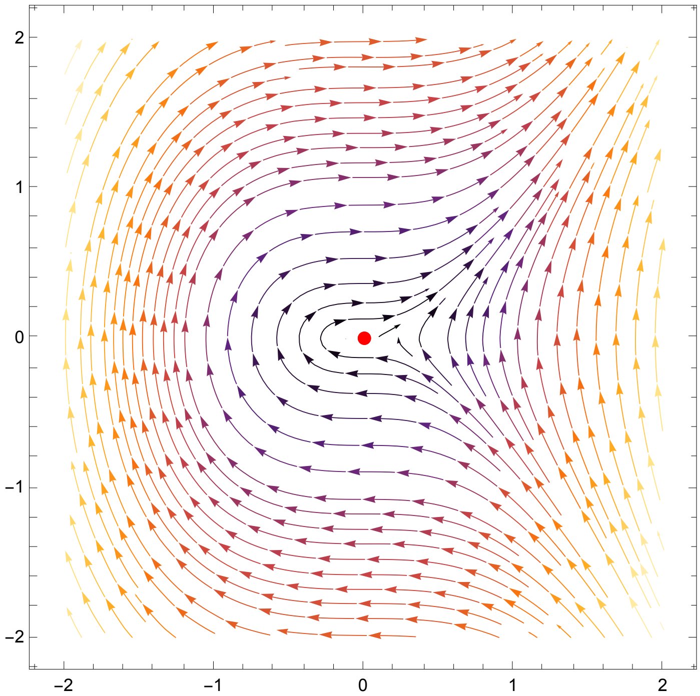
Out[◦]=



In[◦]:=
```
flow4d[x1_, x2_] := {x2, x1^2}
```

In[•]:=
```
Show[StreamPlot[flow4d[x1, x2], {x1, -2, 2}, {x2, -2, 2},
  PlotTheme → "Detailed", StreamColorFunction → "SunsetColors"],
 ListPlot[{Style[{0, 0}, {Red, PointSize → 0.02}]}]]
```

Out[•]=



C24. VS Examples 1

(5) (a) Polar transformation: $r = \sqrt{x_1^2 + x_2^2}$, $\tan\theta = \frac{x_2}{x_1}$

$$\dot{r} = \frac{1}{2r}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$$

$$\dot{\theta} \Rightarrow \frac{d}{dt}\tan\theta = \frac{1}{\cos^2\theta}\dot{\theta} = (1+\tan^2\theta)\dot{\theta} = \left(1+\frac{x_2^2}{x_1^2}\right)\dot{\theta}$$

so then $\frac{r^2}{x_1^2}\dot{\theta} = \frac{d}{dt}\left(\frac{x_2}{x_1}\right) = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2} \Rightarrow \dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$

(b) $\dot{x}_1 = -x_2 + ax_1(x_1^2 + x_2^2)$
$\dot{x}_2 = x_1 + ax_2(x_1^2 + x_2^2)$ $\Rightarrow$ polar

$\dot{x}_1 = -x_2 + ar^2x_1$
$\dot{x}_2 = x_1 + ar^2x_2$ $\Rightarrow$

$$\dot{r} = \frac{-x_1x_2 + ar^2x_1^2 + x_1x_2 + ar^2x_2^2}{r}$$

$$\dot{\theta} = \frac{x_1^2 + ar^2x_1x_2 + x_2^2 - ar^2x_1x_2}{r^2}$$

$$\Rightarrow \boxed{\begin{aligned}\dot{r} &= ar^3 \\ \dot{\theta} &= 1\end{aligned}}$$

$a<0$ stable spiral
$a=0$ centre (nonlinear)
$a>0$ unstable spiral

C24. PS Examples 1

6 (a) Given $\ddot{x} = f(x)$. Let $x_2 = \frac{dx_1}{dt}$ and $x_1 = x$

$$\ddot{x} = f(x) \Rightarrow \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1) \end{aligned}}$$

(b) Given $\ddot{x} = f(x)$ as in (a). Compute

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2 + x_2 f(x_1) = x_2(x_1 + f(x_1)) = \dot{x}_1(x_1 + f(x_1))$$

Therefore $\dot{x}_2 x_2 = \dot{x}_1 f(x_1) \Rightarrow x_2\, dx_2 = f(x_1)\, dx_1$

Integrating, $\boxed{-\int_{x_1^0}^{x_1} f(x_1)\,dx_1 + \frac{1}{2}x_2^2 = \text{constant}}$

The thing on the left is $V(x_1, x_2)$.

(c) $\dot{z}_1 = -z_2 - z_2^3$ Let $z_2 = x_1$ $\Rightarrow$ $\dot{x}_1 = x_2$

$\dot{z}_2 = z_1$ $\qquad z_1 = x_2$ $\qquad \dot{x}_2 = -x_1 - x_1^3$

Thus $\boxed{V(x_1, x_2) = \frac{x_2^2}{2} + \frac{x_1^2}{2} + \frac{x_1^4}{4}}$

Examples 1: Solution for Question 7

Friday, 26 November 2021 12:28

7 (a) $\dot{x} = -y - x^2(x^2+y^2)$

$\dot{y} = x - y^2(x^2+y^2)$

USE $V(x,y) = \frac{1}{2}(x^2+y^2)$:

$\dot{V}(x,y) = x\dot{x} + y\dot{y}$

$= -xy - x^3(x^2+y^2) + xy - xy^2(x^2+y^2)$

$= -(x^2+y^2)$

WE HAVE $\dot{V} < 0 \quad \forall (x,y) \neq (0,0)$

& $V > 0 \quad \forall (x,y) \neq (0,0)$

& $V(0,0) = \dot{V}(0,0) = 0$

& $V(x,y) \to \infty$ AS $x^2+y^2 \to \infty$

$\Rightarrow (x,y) = (0,0)$ IS GLOBALLY ASYMPTOTICALLY STABLE

(b). SYSTEM : $\dot{x} = y$

$\dot{y} = x - x^3 - \gamma y, \quad \gamma > 0$ (DAMPED DUFFING OSCILLATOR)

LET $V(x,y) = 2y^2 - 2x^2 + x^4$

TO MAKE $V(\pm 1, 0) = 0$, ADD A CONSTANT TERM:

$V'(x,y) = 2y^2 + x^4 - 2x^2 + 1 = 2y^2 + (x^2-1)^2$

THEN

$\dot{V}'(x,y) = 4y\dot{y} + 2x.2(x^2-1).\dot{x}$

$= 4yx(1-x^2) - 4\gamma y^2 + 4xy(x^2-1)$

$= -4\gamma y^2$

WE HAVE : $\dot{V}'(x,y) \leq 0 \quad \forall (x,y)$

$V'(x,y) > 0 \quad \forall (x,y) \neq (\pm 1, 0)$

$V'(\pm 1, 0) = 0$

$\Rightarrow (x,y) = (\pm 1, 0)$ IS STABLE

NOTE THAT LYAPUNOV'S METHOD APPLIED TO $V'(x,y)$ DOES <u>NOT</u> SHOW ASYMPTOTIC STABILITY OF $(x,y) = (\pm 1, 0)$

BECAUSE WE HAVE <u>NOT</u> SHOWN THAT $\dot{V}'(x,y) < 0 \ \forall \ (x,y) \neq (\pm 1, 0)$

$\hookrightarrow$ E.G. $\dot{V}'(x,0) = 0 \ \forall \ x$

BUT WE CAN PROVE THAT $(\pm 1, 0)$ IS GLOBALLY ASYMPTOTICALLY STABLE BY APPLYING LASALLE'S INVARIANCE PRINCIPLE USING $V'(x,y)$ — SEE LECTURE 5

(c). SYSTEM: $\dot{x}_1 = -x_1 + 2x_2^3 - 2x_2^4$

$\dot{x}_2 = -x_1 - x_2 + x_1 x_2$

LET $V(x_1, x_2) = x_1^{\alpha_1} + k\, x_2^{\alpha_2}$ WITH $\alpha_1, \alpha_2 \geq 2$, $k > 0$ SO THAT $V \geq 0$

THEN
$$\begin{aligned}
\dot{V}(x_1, x_2) &= \alpha_1 x_1^{\alpha_1 - 1} \dot{x}_1 + k \alpha_2 x_2^{\alpha_2 - 1} \dot{x}_2 \\
&= \alpha_1 x_1^{\alpha_1 - 1}(-x_1 + 2x_2^3 - 2x_2^4) + k\alpha_2 x_2^{\alpha_2 - 1}(-x_1 - x_2 + x_1 x_2) \\
&= -\alpha_1 x_1^{\alpha_1} - k\alpha_2 x_2^{\alpha_2} + (2\alpha_1 x_1^{\alpha_1 - 2} - k\alpha_2 x_2^{\alpha_2 - 4}) \cdot x_1 x_2^3 \\
&\quad + (k\alpha_2 x_2^{\alpha_2 - 4} - 2\alpha_1 x_1^{\alpha_1 - 2}) \cdot x_1 x_2^4
\end{aligned}$$

3RD & 4TH TERMS ARE SIGN-INDEFINITE, SO SET THEM TO 0 BY DEFINING:

$\alpha_1 = 2, \ \alpha_2 = 4 \Rightarrow$ TERMS IN ( ) ARE INDEPENDENT OF x_1, x_2

$k = 1 \Rightarrow$ TERMS IN ( ) ARE ZERO

$\therefore \ V(x_1, x_2) = x_1^2 + x_2^4 \Rightarrow \dot{V}(x_1, x_2) = -2x_1^2 - 4x_2^4$

$$\left.\begin{aligned}
&\text{WE HAVE} \quad \dot{V} < 0 \quad \forall \ (x,y) \neq (0,0) \\
&\& \quad V > 0 \quad \forall \ (x,y) \neq (0,0) \\
&\& \quad V(0,0) = \dot{V}(0,0) = 0 \\
&\& \quad V(x,y) \to \infty \text{ AS } x^2 + y^2 \to \infty
\end{aligned}\right\} \Rightarrow (x,y) = (0,0) \text{ IS } \underline{\text{GLOBALLY ASYMPTOTICALLY STABLE}}$$

C24. DS Examples 1.

(4) (a) $\dot{x} = 2\cos x + \cos y$ Since $\cos(x) = \cos(-x)$ and $\cos(y) = \cos(-y)$

$\dot{y} = 2\cos y + \cos x$ And $\frac{dx}{dt} = \frac{d(-x)}{d(-t)}$ and $\frac{dy}{dt} = \frac{d(-y)}{d(-t)}$

This problem has a symmetry about the origin.

To be conservative, it would have to have orbits. So investigate Jacobian near equilibrium points.

Problem has equilibria when $\cos(x^*) = \cos(y^*) = 0$

$$\underline{x}^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix} + k\begin{bmatrix} \pi \\ 0 \end{bmatrix} + m\begin{bmatrix} 0 \\ \pi \end{bmatrix}$$

Jacobian $\nabla \underline{f} = \begin{bmatrix} -2\sin x & -\sin y \\ -\sin x & -2\sin y \end{bmatrix}$

In particular $\nabla \underline{f}\Big|_{\frac{\pi}{2}, \frac{\pi}{2}} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \Rightarrow \lambda \in \{-3, -1\}$

This is an attracting equilibrium so $\boxed{\text{system is \underline{not} conservative}}$

(b) $\dot{x}_1 = \sin x_2$ This is a gradient system:

$\dot{x}_2 = x_1 \cos x_2$

$$-\frac{\partial V}{\partial x_1} = \sin x_2 \qquad -\frac{\partial V}{\partial x_2} = x_1 \cos x_2$$

Both conditions satisfied if $\boxed{V(x_1, x_2) = -x_1 \sin x_2}$

If we took this as a Hamiltonian the related system would be:

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = -x_1 \cos x_2$$

$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = \sin x_2$$