C24.DS Examples 2

① (a) $\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 + (1 - x_1^2 - x_2^2)x_2$

(i) For stability look at Jacobian matrix

$$\left.\begin{bmatrix} 0 & 1 \\ -1-2x_1x_2 & 1-x_1^2-3x_2^2 \end{bmatrix}\right|_{(@\text{origin})} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

eigenvalues

$-\lambda(1-\lambda)+1=0$

$\lambda^2-\lambda+1=0$

$\lambda = \frac{1 \pm i\sqrt{3}}{2}$ $\quad Re\{\lambda\} > 0$ so origin is UNSTABLE.

(ii) Go to polar coordinates

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} = \frac{x_1x_2 + x_2(-x_1 + (1-r^2)x_2)}{r} = \frac{(1-r^2)x_2^2}{r}$$

$$\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2} = \frac{-x_1^2 + x_1x_2(1-r^2) - x_2^2}{r^2} = \frac{x_1x_2(1-r^2)}{r^2} - 1$$

If $r=1$ then $\dot{r}=0$, $\dot{\theta}=-1$, so there is a limit cycle.

(iii) The limit cycle @ $r=1$ is attractive because $\dot{r}<0$ if $r>1$ and $\dot{r}>0$ if $0<r<1$.

These regions can also be identified as positively invariant:

$1<r_1<\infty$ positively invariant since $\dot{r}<0$ there.

$0<r_0<1$ positively invariant since $\dot{r}>0$ there.

Thus the domain of r such that $r_0<r<r_1$ is positively invariant.

By Poincaré-Bendixson any trajectory starting in this domain has as its ω-limit an equilibrium, a closed orbit, or a finite number of equilibria making up heteroclinic/homoclinic orbits.

Since there are no equilibrium points in the domain, only the closed-orbit option is possible: All trajectories starting anywhere away from the origin will converge to the limit cycle identified in part (ii).

C2HDS Ex 2

1 (b) $\dot{x} = -x^3 + 2y^3$
$\dot{y} = -2xy^2$ has an equilibrium @ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$V(x,y) = \frac{1}{2}(x^2 + y^2)$

$\dot{V} = x\dot{x} + y\dot{y} = -x^4 + 2y^3x - 2y^3x = -x^4 \leq 0$

So the equilibrium is stable.

~~Thus~~ Asymptotic stability cannot be decided w/ Lyapunov because points $[0, y]$ have $\dot{V} = 0$ but aren't equilibria.

Use LaSalle's principle.

The set of points $E = [0, y]$ is such that $\frac{dV}{dt} = 0$

within E the system becomes

$\dot{x} = 2y^3$ Trajectories go right if $y > 0$

$\dot{y} = 0$ left if $y < 0$

So the only trajectory in E that is positively invariant is $(0,0)$.

Therefore $[0,0]$ is an ASYMPTOTICALLY STABLE equilibrium.

C24 DS Examples 2

(2) $\dot{x}_1 = -x_1 + a x_2 + x_1^2 x_2$

$\dot{x}_2 = b - a x_2 - x_1^2 x_2$

$\Rightarrow$ Note $\dot{x}_1 + \dot{x}_2 = b - x_1$

$\dot{x}_1 - \dot{x}_2 = -(x_1 + b) - 2a x_2 - 2 x_1^2 x_2$

So there is a single equilibrium at $(x_1, x_2) = \left(b, \frac{b}{a+b^2}\right)$

(a) Jacobian:

$$\left.\begin{bmatrix} -1 + 2x_1x_2 & a + x_1^2 \\ -2x_1x_2 & -(a+x_1^2) \end{bmatrix}\right|_{\left(b, \frac{b}{a+b^2}\right)} = \begin{bmatrix} \frac{b^2-a}{b^2+a} & a+b^2 \\ -\frac{2b^2}{a+b^2} & -(a+b^2) \end{bmatrix} = \underline{\underline{M}}$$

$\det = -(b^2-a) + 2b^2 = b^2 + a$

$$\text{trace} = \frac{b^2 - a - (a+b^2)^2}{a+b^2} = \frac{-b^4 + (1-2a)b^2 - a(1+a)}{a+b^2}$$

Eigenvalues: $\lambda = \frac{1}{2}\operatorname{tr}\underline{\underline{M}} \pm \frac{1}{2}\sqrt{\operatorname{tr}^2\underline{\underline{M}} - 4\det\underline{\underline{M}}}$

UNSTABLE if $\operatorname{tr}\underline{\underline{M}} > 0$

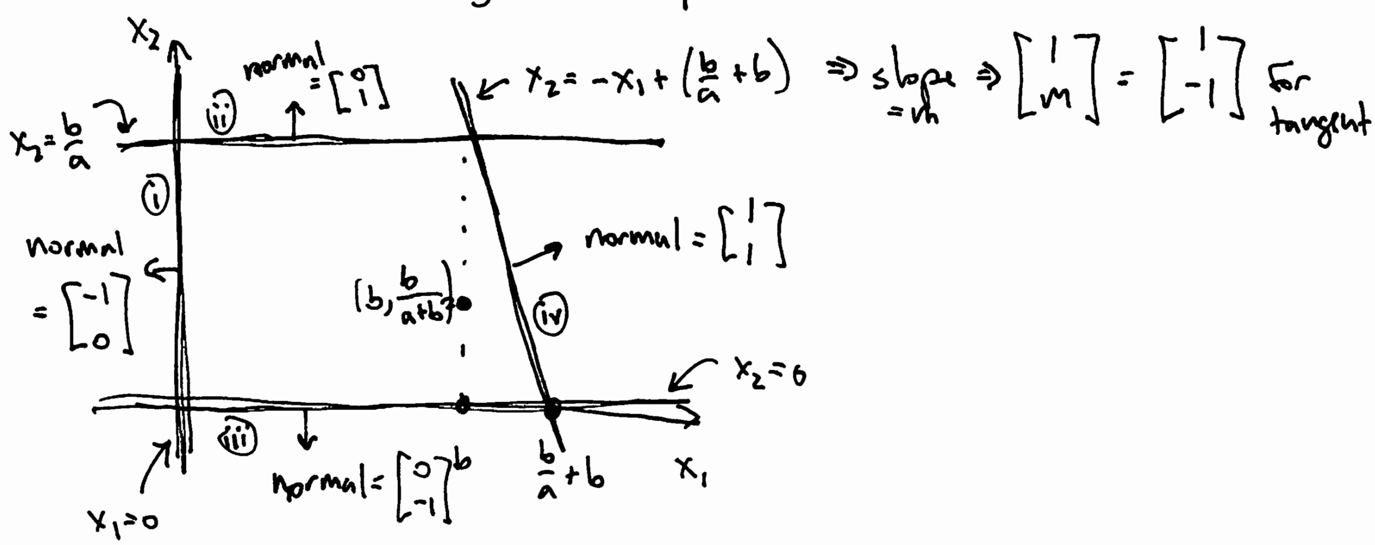
We know that $a > 0, b > 0$ so $\det\underline{\underline{M}} > 0$ always.

Thus we can identify the boundary between stability and instability by finding where $\operatorname{tr}(\underline{\underline{M}}) = 0$. Look as a polynomial in a:

$-a^2 - a(1+2b^2) + b^2(1-b^2) = 0 \Rightarrow a = -\frac{1+2b^2}{2} \pm \sqrt{\frac{1+8b^2}{4}}$ ← only + gives $a > 0$

Domain of instability is
$$\boxed{\begin{array}{l} 0 < b < 1 \\ 0 < a < -\frac{1+2b^2}{2} + \sqrt{\frac{1+8b^2}{4}} \end{array}}$$

(b) First consider the region's shape:



C24 DS Examples 2

(2)(b) cont'd. Take dot products of edge normals with flow

Along (i) $x_1 = 0$ $\Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -\dot{x}_1 = x_1 - ax_2 - x_1^2 x_2$

$\underset{x_1=0}{\Rightarrow} = -ax_2 \leq 0$ for $0 \leq x_2 \leq \frac{b}{a}$

Along (ii) $x_2 = \frac{b}{a}$ $\Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \dot{x}_2 = b - ax_2 - x_1^2 x_2$

$\underset{x_2=\frac{b}{a}}{\Rightarrow} = -\frac{b}{a} x_1^2 \leq 0$ for $0 \leq x_1 \leq b$

Along (iii) $x_2 = 0$ $\Rightarrow \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -\dot{x}_2$

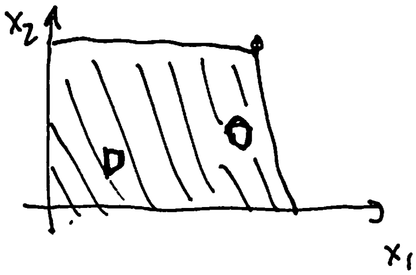
$\underset{x_2=0}{\Rightarrow} = -b \leq 0$ for $0 \leq x_1 \leq \frac{b}{a} + b$

Along (iv) $x_2 = -x_1 + \frac{b}{a} + b$ $\Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \dot{x}_1 + \dot{x}_2 = -x_1 + ax_2 + x_1^2 x_2 + b - ax_2 - x_1^2 x_2$

$= b - x_1 \leq 0$ for $b \leq x_1 \leq \frac{b}{a} + b$

In all cases the dot products are negative: $\boxed{\text{IT IS A TRAPPING REGION}}$

(c) We have an unstable equilibrium point at $(b, \frac{b}{a+b^2})$. All trajectories in the neighborhood of this point must leave it. But no trajectories leave the trapping region from (b). Thus the domain shaded here:



is positively invariant. Since D is positively invariant and contains no equilibria, the Poincaré-Bendixson theorem requires that D contains a closed orbit (limit cycle).

C24 DS Examples 2

(3)(a) $\dot{x}_1 = x_1(2 - x_1 - x_2)$
$\dot{x}_2 = x_2(4x_1 - x_1^2 - 3)$

For Dulac, look at the function
$D(x_1, x_2) = \frac{\partial}{\partial x_1}(B\dot{x}_1) + \frac{\partial}{\partial x_2}(B\dot{x}_2)$

Trying $B = x_1^{-a} x_2^{-b}$ we get that

$$D(x_1, x_2) = \frac{\partial}{\partial x_1}\left[x_1^{1-a} x_2^{-b}(2 - x_1 - x_2)\right] + \frac{\partial}{\partial x_2}\left[x_1^{-a} x_2^{1-b}(4x_1 - x_1^2 - 3)\right]$$
$$= x_1^{-a} x_2^{-b}\left[(1-a)(2 - x_1 - x_2) + (1-b)(4x_1 - x_1^2 - 3) - x_1\right]$$

Let $a = b = 1$ and $D(x_1, x_2) = -\frac{1}{x_2}$, which is negative throughout the first quadrant, so there are <u>no closed orbits</u> there.

To get the four equilibria, set $\dot{x}_1 = \dot{x}_2 = 0$ to get

$$\left.\begin{aligned} 0 &= x_1^*(2 - x_1^* - x_2^*) \\ 0 &= x_2^*(4x_1^* - x_1^{*2} - 3) \end{aligned}\right] \quad \begin{aligned} &\text{True if } x_1^* = 0 \text{ or } (2 - x_1^* - x_2^*) = 0 \\ &x_2^* = 0 \text{ or } (x_1^* - 3)(x_1^* - 1) = 0 \end{aligned}$$

Thus $\boxed{\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$

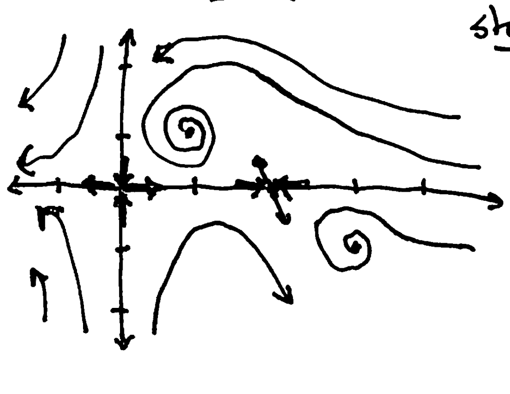
Jacobian? $\underline{\underline{J}} = \begin{bmatrix} 2 - 2x_1 - x_2 & -x_1 \\ -2x_1x_2 + 4x_2 & 4x_1 - x_1^2 - 3 \end{bmatrix}$

$\underline{\underline{J}}(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$
<u>saddle</u>

$\underline{\underline{J}}(2,0) = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$
<u>saddle</u>

$\underline{\underline{J}}(1,1) = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$
$\det(\underline{\underline{J}} - \lambda \underline{\underline{I}}) = \lambda^2 + \lambda + 2$
<u>stable spiral</u>

$\underline{\underline{J}}(3,-1) = \begin{bmatrix} -3 & -3 \\ 2 & 0 \end{bmatrix}$
$\det = \lambda^2 + 3\lambda + 6$
<u>stable spiral</u>



C24 DS Examples 2

③ ⓑ $\dot{x} = y$

$\dot{y} = -ax - by + \alpha x^2 + \beta y^2$

Dulac with $B = be^{-2\beta x}$

$$D(x,y) = \frac{\partial}{\partial x}(By) + \frac{\partial}{\partial y}\left[B(-ax - by + \alpha x^2 + \beta y^2)\right]$$

$$= -2\beta b e^{-2\beta x} y + be^{-2\beta x}(-b + 2\beta y) = -b^2 e^{-2\beta x}$$

Since $D(x,y)$ is negative for any real b, β the system has no limit cycle on the plane.

ⓒ For Bendixson-Dulac examine $D = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2}$

$\dot{x}_1 = -\varepsilon\left(\frac{x_1^3}{3} - x_1 + x_2\right)$

$\dot{x}_2 = -x_1$

$\Rightarrow D = -\varepsilon(x_1^2 - 1)$

Because $\varepsilon > 0$ (given), we see that if $|x_1| < 1$, then $D < 0$. Thus there are no limit cycles contained within this strip of the phase plane. (Note that a limit cycle could pass through this strip, however).

C24 DS Examples 2

(4)(a) Find equilibria

$0 = x_1(4 - x_2 - x_1^2)$ 0 if $x_1 = 0$ or $4 - x_2 - x_1^2 = 0$

$0 = x_2(x_1 - 1)$ 0 if $x_2 = 0$ or $x_1 = 1$

So the equilibria are $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

$\underline{\underline{J}} = \begin{bmatrix} 4 - 3x_1^2 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix} \Rightarrow J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ $J(2,0) = \begin{bmatrix} -8 & -2 \\ 0 & 1 \end{bmatrix}$

$\underline{\text{saddle}}$ $\underline{\text{saddle}}$

$J(-2,0) = \begin{bmatrix} -8 & 2 \\ 0 & -3 \end{bmatrix}$ $J(1,3) = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix} \Rightarrow (-2-\lambda)(-\lambda) + 3 = \lambda^2 + 2\lambda + 3 = 0$

$\lambda = -1 \pm i\sqrt{2}$

$\underline{\text{stable node}}$ $\underline{\text{stable spiral}}$

Saddles have index -1; stable node has index 1; stable spiral has index 1. To get an index of 1 for a stable limit cycle, we could have a closed loop around the stable node $(-2,0)$; a closed loop around the stable spiral $(1,3)$; a closed loop containing the stable spiral, the stable node, and one of the saddles $\{(-2,0),(1,3),(0,0)\}$ or $\{(-2,0),(1,3),(2,0)\}$. So index theory tells us there could be four possible limit cycles.

Now try Dulac with $B = \frac{1}{x_1 x_2}$: We see that

$$D(x_1,x_2) = \frac{1}{x_2}\frac{\partial}{\partial x_1}(4 - x_2 - x_1^2) + \frac{1}{x_1}\frac{\partial}{\partial x_2}(x_1 - 1) = -2\frac{x_1}{x_2}.$$

This doesn't change sign if the trajectory stays in one quadrant. So the closed loop around stable spiral $(1,3)$ cannot be possible. Last note if $x_2 = 0$ $\dot{x}_2 = 0$, and if $x_1 = 0$ $\dot{x}_1 = 0$. So no trajectory can leave the quadrant it starts in. This excludes the other 3 possible limit cycles shown possible by index theory.

C24 DS Examples 2

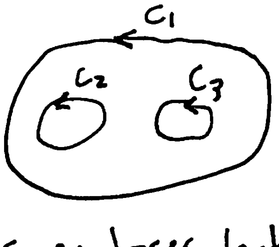
(9)(b) $\dot{x}_1 = x_1 e^{-x_1}$

$\dot{x}_2 = 1 + x_1 + x_2^2$

At equilibrium, the first equation requires $x_1 = 0$. But there is no real x_2 such that $0 = 1 + x_2^2$.

There are no equilibrium points in the phase plane, so the index of any loop will be 0. Thus there is <u>no limit cycle</u>.

(c)



If all are valid closed trajectories, then all have index 1. Individually, C_2 and C_3 must each enclose equilibria whose indices sum to 1. But C_1 encloses both of these sets of equilibria, and together their indices sum to 2. Therefore there must be a set of equilibria inside C_1, but outside both C_2 and C_3, whose indices sum to -1.

It follows that there must be at least one saddle point outside C_2 and C_3 but inside C_1.

(d) $\dot{x} = x^2$

$\dot{y} = -y$

$\underline{\underline{J}} = \begin{bmatrix} 2x & 0 \\ 0 & -1 \end{bmatrix}$. The equilibrium at the origin is a linear centre.

Observe that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a repeat root of the system. Introduce $\varepsilon > 0$, a small parameter:

$\dot{x} = x^2 - \varepsilon$

$\dot{y} = -y$

. This has two equilibria: $\begin{bmatrix} \sqrt{\varepsilon} \\ 0 \end{bmatrix}$ (a saddle) $\begin{bmatrix} -\sqrt{\varepsilon} \\ 0 \end{bmatrix}$ (a stable node)

And $\underline{\underline{J}}$ is unchanged.

Any contour enclosing these equilibria has $\boxed{\text{index } -1+1=0}$. Let $\varepsilon \to 0$.

(e) $\dot{x} = x^2 - y^2 = (x+y)(x-y)$

$\dot{y} = 2xy$

$\underline{\underline{J}} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$. Again the origin is a repeat root. Introduce $\varepsilon > 0$:

$\dot{x} = (x+y)(x-y)$

$\dot{y} = 2xy + 2\varepsilon$

. Equilibria are now $\begin{bmatrix} \sqrt{\varepsilon} \\ -\sqrt{\varepsilon} \end{bmatrix}$ $\begin{bmatrix} -\sqrt{\varepsilon} \\ \sqrt{\varepsilon} \end{bmatrix}$

$\underline{\underline{J}}(-\sqrt{\varepsilon}, \sqrt{\varepsilon}) = 2\sqrt{\varepsilon} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ $(1+\lambda)^2 + 1 \Rightarrow \lambda^2 + \lambda + 1$ <u>stable spiral</u>

$\underline{\underline{J}}(\sqrt{\varepsilon}, -\sqrt{\varepsilon}) = 2\sqrt{\varepsilon} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $(1-\lambda)^2 + 1 \Rightarrow \lambda^2 - 2\lambda + 2$ <u>unstable spiral</u>

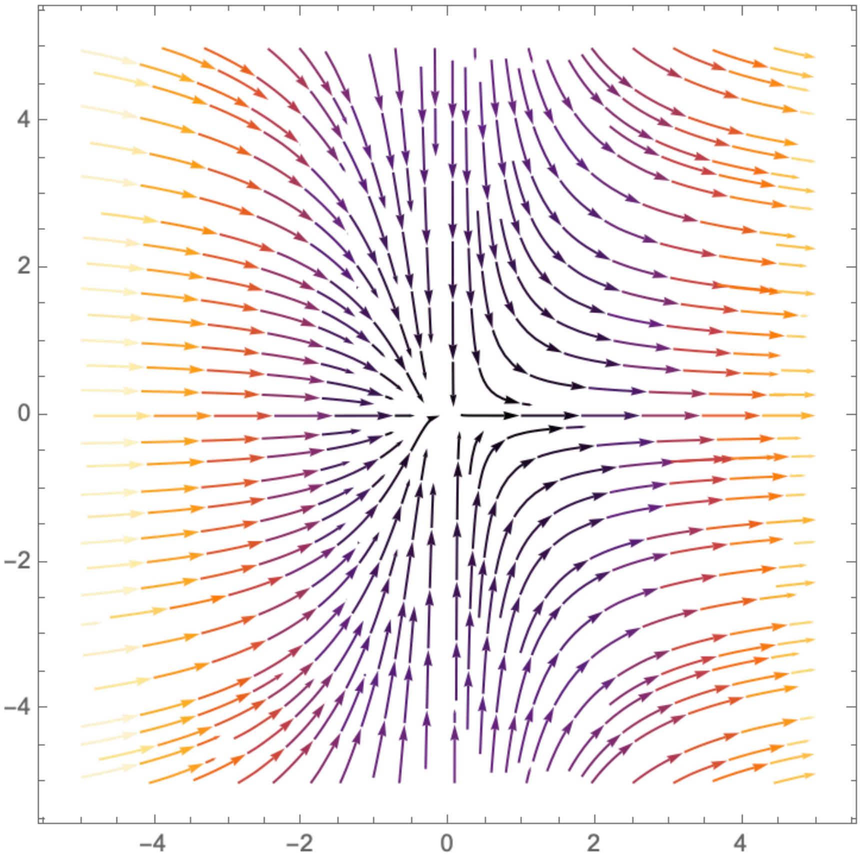
Let $\varepsilon \to 0$

$\boxed{\text{index} = 1 + 1 = 2}$

In[1]:=
```
vector[x_, y_] := {x^2, -y}
```

In[2]:=
```
StreamPlot[vector[x, y], {x, -5, 5}, {y, -5, 5}, PlotTheme → "Detailed",
 StreamPoints → Fine, StreamColorFunction → "SunsetColors"]
```

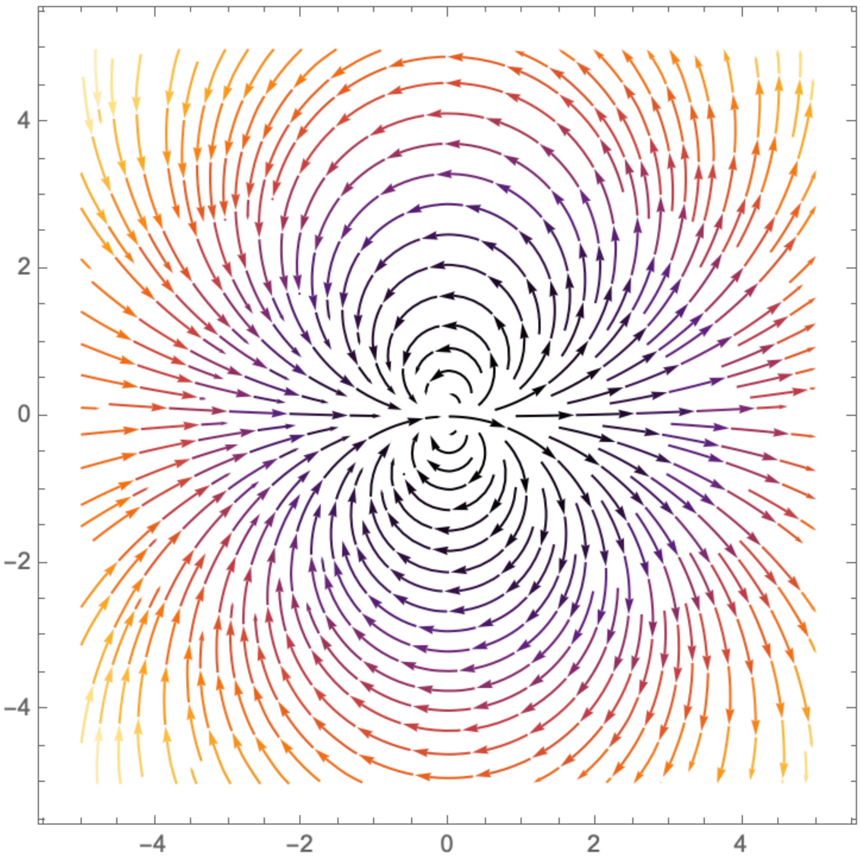
Out[2]=



In[3]:=
```
vector[x_, y_] := {x^2 - y^2, 2 x * y}
```

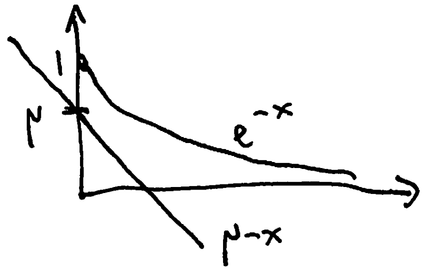
In[4]:=
```
StreamPlot[vector[x, y], {x, -5, 5}, {y, -5, 5}, PlotTheme → "Detailed",
 StreamPoints → Fine, StreamColorFunction → "SunsetColors"]
```

Out[4]=



C24 DS Examples 2

⑤ (a) $f(x;\mu) = \mu - x - e^{-x} = 0$ at equilibrium $\Rightarrow \mu - x = e^{-x}$



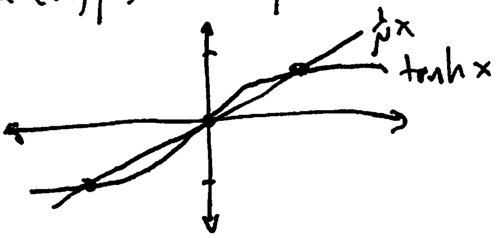
No solutions if $\mu < 1$
Solution @ $x=0$ if $\mu = 1$
Solutions @ positive and negative x if $\mu > 1$.

$$\left.\frac{\partial f}{\partial x}\right|_{0,1} = \left.(-1+e^{-x})\right|_{0,1} = 0 \qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{0,1} = \left.(-e^{-x})\right|_{0,1} = -1 \neq 0$$

$$\left.\frac{\partial f}{\partial \mu}\right|_{0,1} = 1 \neq 0$$

$\boxed{\text{saddle-node bifurcation}}$

(b) $f(x;\mu) = -x + \mu \tanh x = 0$ at equilibrium $\Rightarrow \tanh x = \frac{1}{\mu} x$



If $\mu \leq 1$ one solution $(x=0)$
$\mu > 1$ multiple (3) solutions.

Transition at $\mu = 1$.

$$\left.\frac{\partial f}{\partial x}\right|_{0,1} = \left.\left(-1 + \frac{\mu}{\cosh^2 x}\right)\right|_{0,1} = 0 \qquad \left.\frac{\partial f}{\partial \mu}\right|_{0,1} = \left.\tanh x\right|_{0,1} = 0$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{0,1} = -\left.\frac{2\mu \sinh x}{\cosh^3 x}\right|_{0,1} = 0 \qquad \left.\frac{\partial^2 f}{\partial x \partial \mu}\right|_{0,1} = \left.\frac{1}{\cosh^2 x}\right|_{0,1} = 1 \neq 0$$

$$\left.\frac{\partial^3 f}{\partial x^3}\right|_{0,1} = -2\mu \left[\frac{\cosh^3 x \cdot \cosh x - 3\sinh^2 x \cosh x}{\cosh^6 x}\right]\Bigg|_{0,1} = -2 \neq 0$$

$\boxed{\text{pitchfork bifurcation}}$

(c) $f(x;\mu) = x(\mu + x^3) = 0$ at equilibrium.

Always an equilibrium if $x=0$. As μ goes negative or positive,
the number of equilibria goes from 1 to 2
If $\mu < 0$ equilibrium at $x = \sqrt[3]{-\mu}$ is unstable, $x=0$ stable.
If $\mu > 0$ equilibrium at $x = \sqrt[3]{-\mu}$ is stable, $x=0$ unstable.

$\boxed{\text{A transcritical bifurcation (nonlinear)}}$

C24 DS Examples 2

⑥ $\dot{x}_1 = \mu x_1 - x_2 + \sigma x_1(x_1^2 + x_2^2)$

$\dot{x}_2 = x_1 + \mu x_2 + \sigma x_2(x_1^2 + x_2^2)$

$$\underline{\underline{J}} = \begin{bmatrix} \mu + \sigma(3x_1^2 + x_2^2) & -1 + 2\sigma x_1 x_2 \\ 1 + 2\sigma x_1 x_2 & \mu + \sigma(x_1^2 + 3x_2^2) \end{bmatrix}$$

At the origin, the Jacobian matrix is $\underline{\underline{J}}|_{0,0} = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$

$\det(\underline{\underline{J}}|_{0,0} - \lambda \underline{\underline{I}}) = (\mu - \lambda)^2 + 1 = \lambda^2 - 2\mu\lambda + (\mu^2 + 1)$

Eigenvalues are $\lambda \in \{\mu + i, \mu - i\}$

If $\mu < 0$, stable spiral; $\mu = 0$, linear centre; $\mu > 0$, unstable spiral.

So this is indeed a Hopf bifurcation.

To test criticality, throw this into polar form.

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} = \frac{\mu(x_1^2 + x_2^2) + \sigma(x_1^2 + x_2^2)^2}{r} \Rightarrow \dot{r} = \mu r + 2\sigma r^3$$

$$\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2} = \frac{x_1^2 + x_2^2}{r^2} = 1$$

ⓐ If $\sigma = -1$, then $\dot{r} = r(\mu - 2r^2)$

If $\mu < 0$, then $\dot{r}$ is always negative: stable spiral.

If $\mu = 0$, then $\dot{r}$ is always negative: stable spiral.

If $\mu > 0$, then $\dot{r}$ is positive if $r < \sqrt{\frac{\mu}{2}}$, negative if $r > \sqrt{\frac{\mu}{2}}$

supercritical

ⓑ If $\sigma = 1$, then $\dot{r} = r(\mu + 2r^2)$

If $\mu < 0$, then $\dot{r}$ goes from negative if $r < \sqrt{\frac{\mu}{2}}$ to positive.

If $\mu = 0$, then $\dot{r}$ is positive: unstable spiral.

If $\mu > 0$, then $\dot{r}$ is positive: unstable spiral.

subcritical

ⓒ If $\sigma = 0$, then $\dot{r} = \mu r$.

If $\mu < 0$, then $\dot{r}$ is negative: stable spiral.

If $\mu = 0$, then $\dot{r} = 0$: a centre

If $\mu > 0$, then $\dot{r}$ is positive: unstable spiral.

degenerate

C24 DS Examples 2

7 (a) $x \mapsto x^2 + c$, real quadratic map.

$x^* = x^{*2} + c$ at equilibrium.

$$\boxed{x^* = \frac{1 \pm \sqrt{1-4c}}{2}}$$

For an equilibrium to exist, x^* must be real. So there is only an equilibrium if $\boxed{c \leq \frac{1}{4}}$

(b) To understand stability examine the linearization.

$w_{k+1} = 2x^* w_k \Rightarrow |2x^*| < 1$ for stability.

$|2x^*| = |1 \pm \sqrt{1-4c}|$ Note if $c \leq \frac{1}{4}$ then $1 + \sqrt{1-4c} \geq 1$; unstable.

$|1-\sqrt{1-4c}| < 1 \Rightarrow (1-\sqrt{1-4c})^2 < 1 \Rightarrow 2\sqrt{1-4c} > 1-4c$

$\Rightarrow (1-4c)^2 < 4(1-4c) \Rightarrow |1-4c| < 4 \Rightarrow |\frac{1}{4} - c| < 1$.

Thus $x^* = 1 - \sqrt{1-4c}$ is stable if $-\frac{3}{4} < c < \frac{1}{4}$.

At $c = -\frac{3}{4}$ we have $|2x^*| = 1$ so $x^* = 1 - \sqrt{4} = -1$ is a centre.

We expect a bifurcation at $c = -\frac{3}{4}$.

(c) For a two-cycle $x_{k+2} = f(f(x_k)) = (x_k^2 + c)^2 + c = x_k^4 + 2cx_k^2 + c^2 + c$.

This will reach equilibrium when $x_2^* = x_2^{*4} + 2cx_2^{*2} + c^2 + c$.

We want $x_2^{*4} + 2cx_2^{*2} - x_2^* + c^2 + c = 0$. We can solve this, because we know that x^* from above are also roots of this expression.

$$x_2^{*4} + 2cx_2^{*2} - x_2^* + c^2 + c = \left(x_2^* - \frac{1+\sqrt{1-4c}}{2}\right)\left(x_2^* - \frac{1-\sqrt{1-4c}}{2}\right)(ax_2^{*2} + bx_2^* + d)$$

$$= (c - x_2^* + x_2^{*2})(ax_2^{*2} + bx_2^* + d)$$

$$= ax_2^{*4} + (b-a)x_2^{*3} + (-b + ac + d)x_2^{*2} + (bc - d)x_2^* + cd$$

We conclude that $a = 1$, $b = 1$, $d = c+1$, completing the factorization:

$$= (c - x_2^* + x_2^{*2})(x_2^{*2} + x_2^* + 1 + c) = 0$$

$$\Rightarrow \boxed{x_2^* = \frac{-1 \pm \sqrt{-3-4c}}{2}}$$ is the fixed point of a 2-cycle.

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(7) (c) cont'd. Once again examine stability through the linearization.

$$w_{k+2} = (4x_2^{*3} + 4cx_2^*)w_k = 4x_2^*(x_2^{*2} + c)w_k.$$

Stability requires that $|4x_2^*(x_2^{*2}+c)| < 1$.

Now $x_2^{*2} = \frac{1}{4}(-1 \pm \sqrt{-3-4c})^2 = \frac{1}{4}(1 \mp 2\sqrt{-3-4c} - 3 - 4c) = -\frac{1}{2} - c \mp \frac{1}{2}\sqrt{-3-4c}$

So $4x_2^*(x_2^{*2}+c) = (-1 \pm \sqrt{-3-4c})(-1 \mp \sqrt{-3-4c}) = 1-(-3-4c) = 4(1+c)$

The 2-cycle equilibria will therefore be stable if $|4(1+c)| < 1$.

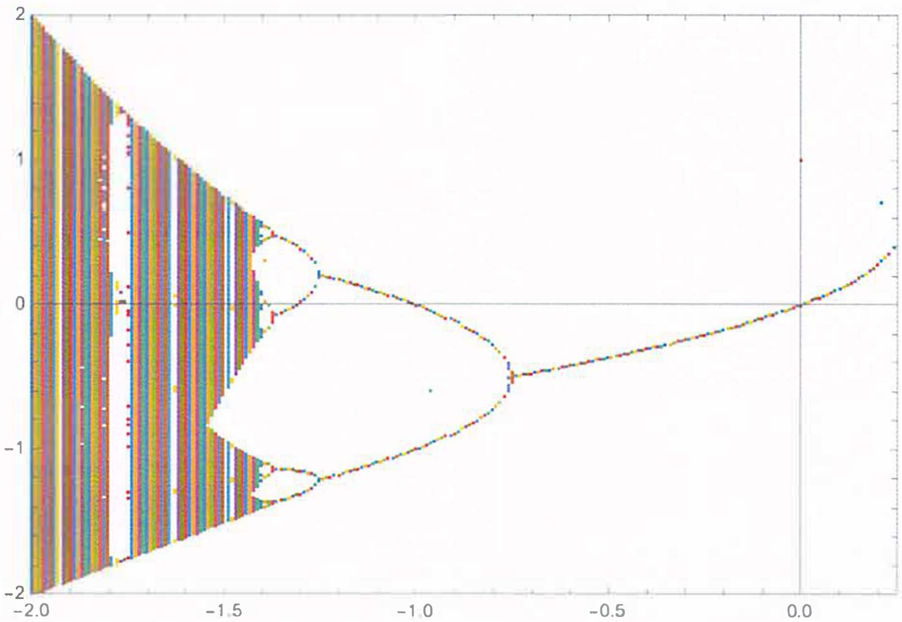
$|1+c| < \frac{1}{4}$ if $\boxed{-\frac{5}{4} < c < -\frac{3}{4}}$.

(d) See attached Mathematica plot.

```
In[1]:= (* Orbit diagram for problem 7d. *)

In[2]:= ListPlot[Table[Thread[{r, Nest[#^2 + r &, Range[0, 1, 0.0005], 500]}],
   {r, -2, 1/4 - 0.001, 0.01}], Frame → True, AspectRatio → 2/3,
  PlotRange → {{-2, 1/4}, {-2, 2}}]
```

Out[2]=



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⑧ (a) Test $\vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}(-\mu x + zy) + \frac{\partial}{\partial y}[-\mu y + (z-a)x] + \frac{\partial}{\partial z}(1-xy)$

$$= -\mu - \mu = -2\mu$$

Thus if μ is positive the system is dissipative.

(b) $\dot{x} = -\mu x + zy$

$\dot{y} = -\mu y + (z-a)x$ at a fixed point $\mu x = zy$ (i), $\mu y = (z-a)x$ (ii), $xy = 1$ (iii)

$\dot{z} = 1 - xy$

Equate (i)$\cdot y$ with (ii)$\cdot x$ $\Rightarrow zy^2 = (z-a)x^2 \Rightarrow z = \dfrac{ax^2}{x^2-y^2}$.

But by (iii) $y = \frac{1}{x}$ so $z = \dfrac{ax^4}{x^4-1}$. (iv)

Now put (iii) in (i) to get $z = \mu x^2$ and insert (iv): $x^4 - \frac{a}{\mu}x^2 - 1 = 0$.

Thus $x^2 = \frac{a}{2\mu} \pm \sqrt{1 + \left(\frac{a}{2\mu}\right)^2}$. Has to be positive so only + works.

Therefore $\boxed{x = \pm\sqrt{\frac{a}{2\mu} + \sqrt{1+\left(\frac{a}{2\mu}\right)^2}}}$

Let $\boxed{k = \sqrt{\frac{a}{2\mu} + \sqrt{1+\left(\frac{a}{2\mu}\right)^2}}}$. Then $z = \dfrac{ak^4}{k^4-1}$ simplifies:

$$z = \frac{a}{1-\frac{1}{k^4}} = \frac{ak^4}{\left(\frac{a}{2\mu}\right)^2 + 2\frac{a}{2\mu}\sqrt{1+\left(\frac{a}{2\mu}\right)^2} + \left(\frac{a}{2\mu}\right)^2} = \frac{\mu k^4}{\frac{a}{2\mu} + \sqrt{1+\left(\frac{a}{2\mu}\right)^2}} = \mu k^2$$

Thus $\boxed{x^* = \pm k,\ y^* = \pm\frac{1}{k},\ z^* = \mu k^2, \text{ with } k = \sqrt{\frac{a}{2\mu} + \sqrt{1+\left(\frac{a}{2\mu}\right)^2}}}$

We can check to see $a = \mu\left(k^2 - \frac{1}{k^2}\right)$, as suggested.

If $x^* = \pm k$, $y^* = \pm\frac{1}{k}$ then (i) $\Rightarrow z = \mu k^2$ (ii) $\Rightarrow \frac{\mu}{k^2} = z - a$

and consequently $a = \mu\left(k^2 - \frac{1}{k^2}\right)$, as we expected.

C24 DS Examples 2

⑧ cont'd © To classify stability examine the Jacobian:

$$\underline{\underline{J}} = \begin{bmatrix} -\mu & z & y \\ z-a & -\mu & x \\ -y & -x & 0 \end{bmatrix}$$ Find eigenvalues, $\det(\underline{\underline{J}}|_{x^*,y^*,z^*} - \lambda \underline{\underline{I}}) = 0$.

$$\det(\underline{\underline{J}} - \lambda\underline{\underline{I}}) = -(\mu+\lambda)\begin{vmatrix} -(\mu+\lambda) & x \\ -x & -\lambda \end{vmatrix} - z\begin{vmatrix} z-a & x \\ -y & -\lambda \end{vmatrix} + y\begin{vmatrix} z-a & -(\mu+\lambda) \\ -y & -x \end{vmatrix}$$

$$= -(\mu+\lambda)^2\lambda - (\mu+\lambda)x^2 + z(z-a)\lambda - xyz - xy(z-a) - y^2(\mu+\lambda)$$

$$= -\mu^2\lambda - 2\mu\lambda^2 - \lambda^3 - x^2\lambda - \mu x^2 + z^2\lambda - az\lambda + axy - 2xyz - \mu y^2 - y^2\lambda$$

$$= -\lambda^3 - 2\mu\lambda^2 + (z^2 - \mu^2 - x^2 - y^2 - az)\lambda + (axy - 2xyz - \mu x^2 - \mu y^2).$$

When evaluated at either equilibrium point

$$\det(\underline{\underline{J}}|_{x^*,y^*,z^*} - \lambda\underline{\underline{I}}) = -\lambda^3 - 2\mu\lambda^2 + (\mu^2k^4 - \mu^2 - k^2 - \tfrac{1}{k^2} - a\mu k^2)\lambda + (a - 3\mu k^2 - \tfrac{\mu}{k^2})$$

and since, as we just showed, $a = \mu(k^2 - \frac{1}{k^2})$,

$$= -\lambda^3 - 2\mu\lambda^2 + [\mu^2k^4 - \mu^2 - k^2 - \tfrac{1}{k^2} - \mu k^2(\mu k^2 - \tfrac{\mu}{k^2})]\lambda + (\mu k^2 - \tfrac{\mu}{k^2} - 3\mu k^2 - \tfrac{\mu}{k^2})$$

$$= -\lambda^3 - 2\mu\lambda^2 - (k^2 + \tfrac{1}{k^2})\lambda + 2\mu(k^2 + \tfrac{1}{k^2})$$

Let $b = k^2 + \frac{1}{k^2} = \dfrac{[\frac{a}{2\mu} + \sqrt{1+(\frac{a}{2\mu})^2}]^2 + 1}{\frac{a}{2\mu} + \sqrt{1+(\frac{a}{2\mu})^2}} = \dfrac{2[1+(\frac{a}{2\mu})^2] + 2\frac{a}{2\mu}\sqrt{1+(\frac{a}{2\mu})^2}}{\frac{a}{2\mu} + \sqrt{1+(\frac{a}{2\mu})^2}}$

That is, $b = 2\sqrt{1+(\frac{a}{2\mu})^2}$; then the characteristic polynomial is

$$\det(\underline{\underline{J}}|_{x^*,y^*,z^*} - \lambda\underline{\underline{I}}) = -\lambda^3 - 2\mu\lambda^2 - b\lambda + 2\mu b = -\lambda(\lambda^2+b) - 2\mu(\lambda^2+b)$$

$$= -(\lambda+2\mu)(\lambda^2+b) = 0.$$

This has roots $\lambda \in \{-2\mu, i\sqrt{2\sqrt{1+(\frac{a}{2\mu})^2}}, -i\sqrt{2\sqrt{1+(\frac{a}{2\mu})^2}}\}$.

Thus there is a stable subspace, corresponding to the negative real eigenvalue, and a centre subspace, corresponding to the pair of imaginary eigenvalues.

```
In[1]:= (* Simulation of Rikitake system for problem 8d. *)

In[2]:= μ := 2

In[3]:= a := 5

In[4]:= ParametricPlot3D[
         Evaluate[{x[t], y[t], z[t]} /. NDSolve[{x'[t] == -μ * x[t] + z[t] * y[t],
             y'[t] == -μ * y[t] + (z[t] - a) * x[t], z'[t] == 1 - x[t] * y[t], x[0] == 0.1,
             y[0] == 0.0, z[0] == 0.0}, {x, y, z}, {t, 0, 200}, MaxSteps → Infinity]],
         {t, 0, 200}, PlotPoints → 5000, PlotStyle → {Thin, Blue},
         AxesLabel → {x, y, z}]
```

Out[4]=