A10613W1

TRINITY TERM 2016

SECOND PUBLIC EXAMINATION

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (PAPER C24)

Honour School of Engineering Science

Monday 13 June 2016 14:30 – 16:00

Answers to not more than **THREE** *questions should be submitted, and each question must be answered in a separate booklet.*

DO NOT TURN THIS PAGE UNTIL TOLD YOU MAY DO SO

Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*

1. (a) State the Hartman-Grobman theorem and explain what it means.

[4 marks]

(b) Consider a dynamical system of the form

$$\ddot{x} + \sin x = 0$$

(i) Transform this system into a system of first order equations.

(ii) Show that this system is a Hamiltonian system by giving an expression for the Hamiltonian function.

(iii) Characterise the linear stability properties of all equilibria of this system and sketch the phase portrait. Using the Hartman-Grobman theorem, what can you conclude for the stability properties of the equilibria for the nonlinear system?

(iv) Use Lyapunov's theorem to show that the equilibrium point $(0, 0)$ is stable. Hint: Consider the Hamiltonian function as candidate for a Lyapunov function.

[8 marks]

(c) Consider the modified dynamical system of the form

$$\ddot{x} + \sin x = \varepsilon \dot{x}.$$

with $\varepsilon < 0$. Use LaSalle's invariance principle to show that the equilibrium $(0, 0)$ of this system is asymptotically stable.

[4 marks]

2. (a) State the conditions that have to be satisfied for a system undergoing a Hopf bifurcation at $\mu = \mu_0$. Describe what happens for a supercritical and a subcritical Hopf bifurcation at values ranging from $\mu < \mu_0$ to $\mu > \mu_0$ and sketch the phase portraits for $\mu < \mu_0$ and $\mu > \mu_0$.

[5 marks]

(b) Consider a dynamical system of the form

$$\dot{x}_1 = x_1(\mu - x_1^2 - x_2^2) - x_2$$
$$\dot{x}_2 = x_2(\mu - x_1^2 - x_2^2) + x_1.$$

Characterise the linear stability properties of the equilibrium $(0, 0)$ as a function of μ.

[4 marks]

(c) (i) Show that a two-dimensional nonlinear system with states x_1 and x_2 can be written in polar coordinates using

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$$
$$\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$$

(ii) Transform the system in (b) into polar coordinates (r, θ).

(iii) Consider the system in polar coordinates for $\mu > 0$. Show that this system has a limit cycle, find it and determine its stabilty.

(iv) Conclude from your results what kind of bifurcation the system undergoes at $\mu = 0$.

[7 marks]

3. Consider the following differential equation:

$$\varepsilon^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

where $y(0) = a$ and $y(1) = 2$.

(a) Show, using a perturbation series, that there is an exact solution if $a = 2$, but that there must be a boundary layer otherwise.

[6 marks]

(b) Now take $a = 1$. Assuming that the boundary layer is located at the origin, use a boundary layer co-ordinate ($\bar{x}$) to transform the differential equation into the form:

$$\frac{d^2Y}{d\bar{x}^2} - \bar{x}\frac{dY}{d\bar{x}} = 0$$

and, by matching the two solutions, show that the solution of the equation is:

$$y = 1 + \frac{\int_0^{\bar{x}} e^{\bar{x}^2/2} d\bar{x}}{\int_0^{\infty} e^{\bar{x}^2/2} d\bar{x}}$$

[10 marks]

4. (a) What are the conditions on a function $K(x, x')$ that are required for it to be a valid covariance function?

[4 marks]

(b) Let $K_1(x, x')$ and $K_2(x, x')$ be valid covariance functions. What conditions on $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are required for

$$K_3(x, x') = \alpha K_1(x, x') + \beta K_2(x, x')$$

to be a valid covariance function?

[6 marks]

(c) Assuming the data in Figure 1 below is to be modelled with a Gaussian process, write an appropriate covariance function $K(x, x')$, and explain your reasoning.

[6 marks]

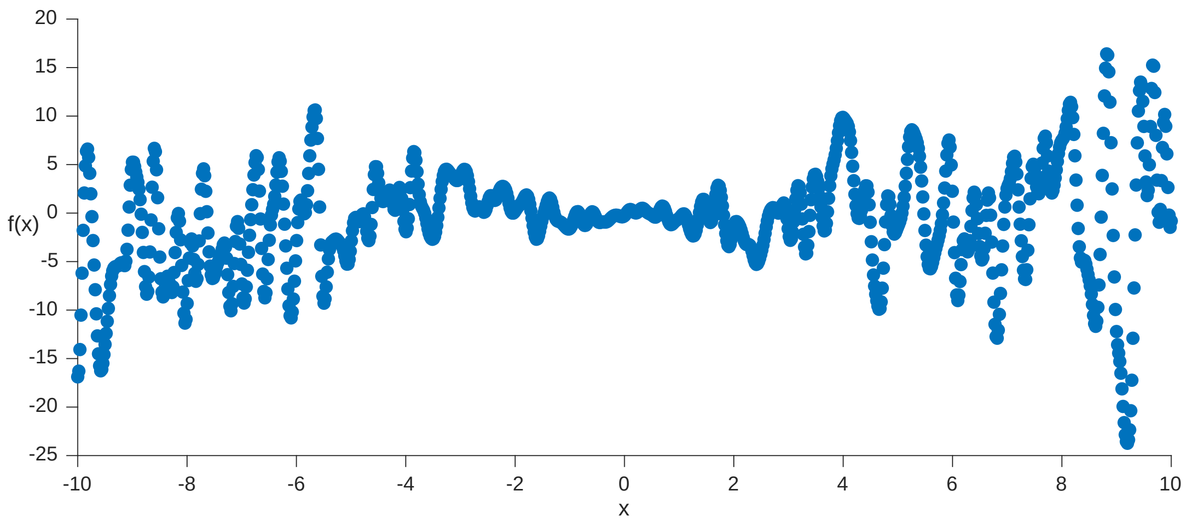

Figure 1