C 24: Dynamical Systems

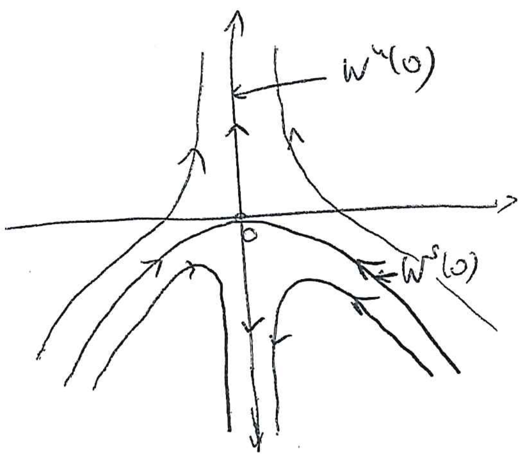
Answers to Examination Questions 2016

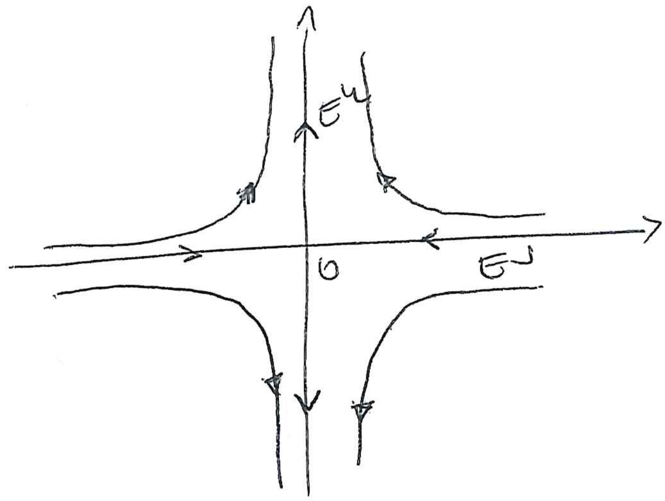
Question 1

1(a) In the case of hyperbolic equilibria, there exists a bi-continuous function H of a region containing the equilibrium (of the phase space of the nonlinear system) to a region containing the origin (of the phase-space of the linearisation) such that trajectories are mapped exactly, and parametrisation of time is preserved.

Meaning:
- Near the equilibrium, the stable linear subspace is mapped to a stable manifold in a region surrounding the equilibrium point
- Near the equilibrium, the unstable linear subspace is mapped to an unstable manifold in a region surrounding the equilibrium point
- Nothing is said about centres.

Example:





(picture not compulsive)

1.(b) $\ddot{x} + \sin x = 0$

(i) $x_1 = x$, $x_2 = \dot{x}$ $\Rightarrow$ $\dot{x}_1 = x_2$, $\dot{x}_2 = -\sin x_1$

(ii) $H(x_1, x_2) = \frac{1}{2} x_2^2 - \cos x_1$

$$\longrightarrow \quad \dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2$$
$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\sin x_1$$

(iii) equilibria : $x^* = (k \cdot \pi, 0)$, $k = 0, 1, 2, \ldots$
$-1, -2, \ldots$

Jacobian

$$J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{pmatrix}$$

Distinguish between $x^* = (k\pi, 0)$, $k = \ldots -2, 0, 2, 4 \ldots$
and $\tilde{x} = (k \cdot \pi, 0)$, $k = \ldots -3, -1, 1, 3, \ldots$

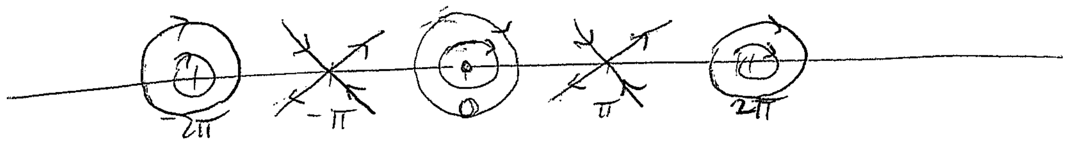
$$\Rightarrow J(x^*) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenvalues: of $J(x^*)$: $\lambda_{1,2} = \pm i$ (non-hyperbolic)
of $J(\tilde{x})$: $\lambda_{1,2} = \pm 1$ (hyperbolic)

$\Rightarrow$ x^* is linear centre, $\tilde{x}$ is saddle (unstable)

eigenvectors: x^*, $\lambda_1 = i$, $v_1 = \begin{pmatrix} -1 \\ i \end{pmatrix}$
$\lambda_2 = -i$, $v_2 = \begin{pmatrix} -1 \\ -i \end{pmatrix}$

$\tilde{x}$, $\lambda_1 = -1$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\Rightarrow E^s = \{(1, -1)\}$
$\lambda_2 = +1$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow E^u = \{(1, 1)\}$



by Hartman-Grobman : $\tilde{x}$ also non-linear saddle, but we do not know about x^* (because non-hyperbolic)

(iv) A natural Lyapunov function candidate is the Hamiltonian function

$$\bar{V}(x_1,x_2) = \frac{1}{2}x_2^2 - \cos x_1$$

However, $\bar{V}(0,0) = -1$

$\Rightarrow$ choose $V(x_1,x_2) = \frac{1}{2}x_2^2 - \cos x_1 + 1$

$\Rightarrow$ $V(0,0) = 0$ and $V(x_1,x_2) > 0$ for $(x_1,x_2) \neq (0,0)$

$\Rightarrow$ condition (i) is satisfied

The derivative of $V(x_1,x_2)$ along the trajectory of the system is given by

$$\dot{V}(x_1,x_2) = \nabla V(x_1,x_2) \cdot f(x) =$$

$$x_2\dot{x}_2 + \sin x_1 \dot{x}_1 =$$

$$x_2(-\sin x_1) + \sin x_1 \cdot x_2 = 0$$

$\Rightarrow$ condition (ii) is satisfied

$\Rightarrow$ $(0,0)$ is stable

1. (e) first order system $\dot{x}_1 = x_2$
$\dot{x}_2 = -\sin x_1 + \varepsilon x_2$

Consider function $V(x_1,x_2) = \frac{1}{2}x_2^2 - \cos x_1 + 1$

$\Rightarrow$ $\dot{V}(x_1,x_2) = -\varepsilon x_2^2 \leq 0$ is negative everywhere except on the line $x_2 = 0$, where $\dot{V}(x_1,x_2) = 0$

$\Rightarrow$ set E consists of all points with $x_2 = 0$, but only the points $x_1 = k\cdot\pi$, $k \in \mathbb{Z}$ are invariant.

The level set $V = c$ with $0 < c < \pi$ is the boundary of a positively invariant region and that region contains only the equilibrium point $(0,0)$ $\Rightarrow$ this defines the positively invariant set D.

$\Rightarrow$ $E = \{x \in D$ such that $\dot{V}(x) = 0\} = \{x \in D, x_2 = 0\}$

$\Rightarrow$ $M = \{(0,0)\}$ $\xRightarrow{\text{Lasalle}}$ all trajectories starting at $x \in D$ tend to M as $t \to \infty$ $\Rightarrow$ $(0,0)$ is asym. stable

Question 2

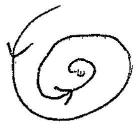
2(a). 1. For $\mu < \mu_0 + \varepsilon$ and $\mu > \mu_0 - \varepsilon$ for $\varepsilon > 0$

$$\lambda_{12}(\mu) = \alpha(\mu) \pm j\omega(\mu)$$

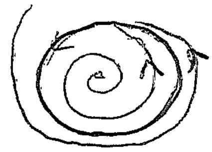
2. $\alpha(\mu_0) = 0$
3. $\alpha(\mu) < 0$ for $\mu < \mu_0$
4. $\alpha(\mu) > 0$ for $\mu > \mu_0$

Supercritical Hopf bifurcation:

For $\mu < \mu_0$ we have a stable spiral fixed point, which for $\mu > \mu_0$ becomes an unstable spiral. The unstable spiral is bounded by a stable limit cycle which expands with increasing μ



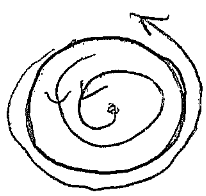
$\mu < \mu_0$



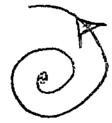
$\mu > \mu_0$

Subcritical Hopf bifurcation:

For $\mu < \mu_0$ a stable spiral is surrounded by an unstable limit cycle. As μ increases, the unstable limit cycle becomes smaller and at $\mu = \mu_0$ the cycle collapses on a fixed point, which for $\mu > \mu_0$ behaves as an unstable spiral.



$\mu < \mu_0$



$\mu > \mu_0$

(2)(b) $\dot{x}_1 = x_1(\mu - x_1^2 - x_2^2) - x_2$

$\dot{x}_2 = x_2(\mu - x_1^2 - x_2^2) + x_1$

linearisation in $(0,0)$: $J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$

eigenvalues : $\lambda_{1,2} = \mu \pm i$

$\mu < 0$: $(0,0)$ is stable spiral

$\mu = 0$: $(0,0)$ is centre

$\mu > 0$: $(0,0)$ is unstable spiral

2(c) (ii) polar coordinates :

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$$

$$= \frac{x_1(x_1(\mu - x_1^2 - x_2^2) - x_2) + x_2(x_2(\mu - x_1^2 - x_2^2) + x_1)}{r}$$

$$= \frac{(x_1^2 + x_2^2)(\mu - x_1^2 - x_2^2)}{r} \qquad (r^2 = x_1^2 + x_2^2)$$

$$= \mu r - r^3$$

$$\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$$

$$= \frac{x_1(x_2(\mu - x_1^2 - x_2^2) + x_1) - x_2(x_1(\mu - x_1^2 - x_2^2) - x_2)}{r^2}$$

$$= \frac{x_1^2 + x_2^2}{r^2} = 1$$

(i) Since $r^2 = x_1^2 + x_2^2$

Differentiating gives $r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$

$\Rightarrow \dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$

Similarly, $\tan\theta = \frac{x_2}{x_1}$

$\Rightarrow (1 + \tan^2\theta)\dot{\theta} = \frac{-\dot{x}_1 x_2 + \dot{x}_2 x_1}{x_1^2}$

$\Rightarrow \boxed{\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2(1 + x_2^2/x_1^2)} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}}$

(iii) $\dot{r} = \mu r - r^3 \quad , \mu > 0$

$\dot{\theta} = 1$

$r = \sqrt{\mu} \quad : \quad \dot{r} = 0 \quad \Rightarrow$ limit cycle

$$\left.\begin{array}{lll} r < \sqrt{\mu} & : & \dot{r} > 0 \\ r > \sqrt{\mu} & : & \dot{r} < 0 \end{array}\right\} \text{limit cycle is stable}$$

(iv) Since there is a stable limit cycle for $\mu > \mu_0 = 0$ we have a supercritical Hopf bifurcation.