A10613W1

TRINITY TERM 2017

SECOND PUBLIC EXAMINATION

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (PAPER C24)

Honour School of Engineering Science
Honour School of Engineering, Economics and Management

Thursday 08 June 2017 09:30 – 11:00

Answers to not more than **THREE** *questions should be submitted, and each question must be answered in a separate booklet.*

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Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*

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1. A two-dimensional dynamical system can have a number of equilibrium points. One of these points, when linearised, is found to be a centre.

(a) Describe a 'centre' and what the Hartman-Grobman theorem has to say about such equilibrium points. As part of your explanation describe a 'hyperbolic equilibrium point'.

[2 marks]

(b) Explain how transformation from Cartesian to polar coordinates can sometimes resolve uncertainties about the nature of an equilibrium point and derive the equations that perform the transformation.

[7 marks]

(c) Consider the dynamical system

$$\begin{aligned}\dot{x}_1 &= -x_2 + \mu x_1\left(x_1^2 + x_2^2\right)\\ \dot{x}_2 &= x_1 + \mu x_2\left(x_1^2 + x_2^2\right)\end{aligned}$$

(i) Show that the origin (0,0) is an equilibrium point and that the eigenvalues are independent of μ.

[2 marks]

(ii) Change to polar coordinates and characterise the stability properties of the origin for $\mu < 0$ and $\mu > 0$.

[4 marks]

(iii) What happens at $\mu = 0$?

[1 mark]

2. (a) Describe how a Poincaré map is constructed for the case of a suspected limit cycle in a two-dimensional dynamical system.

[2 marks]

(b) Describe how a cobweb diagram may be used to determine the stability properties of the limit cycle found using a Poincaré map.

[2 marks]

(c) A two-dimensional dynamical system has the following polar coordinate definition;

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r\left(1-r^2\right)}{4}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

(i) Does the system possess a periodic orbit? If so, what is its radius?

[2 marks]

(ii) The time taken for one cycle around the origin is 2π seconds. Consider the Poincaré map generated by the line $\theta = \theta_1$. If the trajectory is initially at $r = r_1$ when $\theta = \theta_1$ and subsequently at $r = r_2$ when $\theta = \theta_1 + 2\pi$, explain why

$$\int_{r_1}^{r_2} \frac{4\mathrm{d}r}{r\left(1-r^2\right)} = 2\pi.$$

[2 marks]

(iii) Use the integral above to show how a cobweb map may be constructed using the functions

$$r_{k+1} = f(r) = \frac{1}{\sqrt{1 + e^{-\pi}\left(r_k^{-2} - 1\right)}}$$

and

$$r = r_{k+1}.$$

Sketch the resulting cobweb diagram and demonstrate that the limit cycle is stable.

[8 marks]

3. (a) Consider the simple quadratic equation

$$\varepsilon m^2 + m(1 + \varepsilon) + 1 = 0$$

Find the solutions to this equation and explain why this can be considered as a singular perturbation problem as $\varepsilon \to 0$.

[4 marks]

(b) An approximate solution is sought for the second-order ordinary differential equation

$$\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx}(1 + \varepsilon) + y = 0$$

for $0 < \varepsilon \ll 1$, over the region $0 \leq x \leq 2$.

The boundary conditions on the ODE are $y = 1$ at both $x = 0$ and $x = 2$.

(i) Find the form of the lowest order (in ε) approximation to the solution of the ODE, and show that this cannot satisfy both boundary conditions.

(ii) The next order contribution to the solution can be obtained from the first two terms in the ODE. Show that this has the form of a simple boundary layer and explain why it can only be located at one of the boundaries.

(iii) Using asymptotic matching, find the lowest order composite approximation for the solution which is approximately valid everywhere.

[10 marks]

(c) Consider the same ODE and boundary conditions but now with $0 < -\varepsilon \ll 1$. Without solving the equation, explain why the position of the boundary layer must move.

[2 marks]

4. (a) With reference to Bayes' rule and defining all the terms, describe the principal mechanism of Bayesian model selection. What is its principal adavantage? Explain why, in practice, often only the *evidence* (or *model likelihood*) is used to evaluate the model fit.

[4 marks]

(b) A particular *sequence* of outcomes $\mathcal{D}$ of N tosses of a coin yields k consecutive *heads* followed by $q \triangleq N - k$ consecutive *tails*. Let $p(head) = \theta$ be the probability that an individual coin toss yields the outcome *head*. Further, let $p(\theta)$ follow a Beta distribution such that

$$\begin{aligned} \theta &\sim Beta(\theta; a, b) \\ &= \frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1} \end{aligned}$$

where $B(a, b)$ denotes the *Beta function* as defined below.

(i) For the specific sequence of outcomes $\mathcal{D}$ described above, show that the *evidence* $p(\mathcal{D} \mid M)$ can be expressed as the ratio of two *Beta functions*, such that

$$p(\mathcal{D} \mid M) = \frac{B(a+k, b+q)}{B(a,b)}$$

(ii) Consider two candidate models representing symmetric but opposite biases: model M_1 where $\theta \sim Beta(\theta; 4, 6)$ (see Figure 1(a)); and model M_2 where $\theta \sim Beta(\theta; 6, 4)$ (see Figure 1(b)). Show that the ratio of the model evidence terms (also known as the *Bayes factor*) for both of these models is given by

$$\frac{p(\mathcal{D} \mid M_1)}{p(\mathcal{D} \mid M_2)} = \frac{(q+4)(q+5)}{(k+4)(k+5)} \tag{1}$$

[9 marks]

(c) For the sequences observed in each case below, evaluate the Bayes factor (Equation 1) and, in each case, interpret your answer.

(i) 5 heads followed by 5 tails;
(ii) 3 heads followed by 7 tails;
(iii) 7 heads followed by 3 tails.

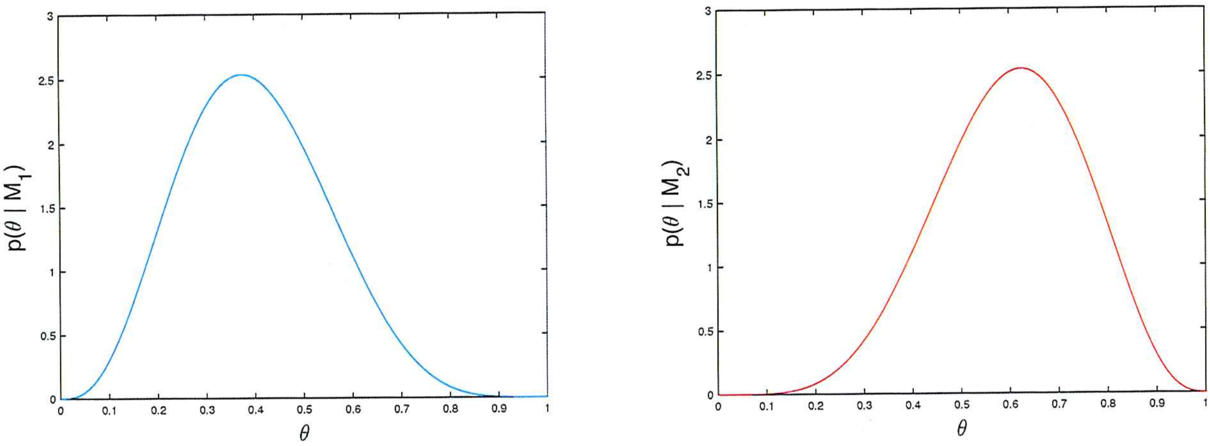
HINT: You may find it useful to know that the *Beta function* used in the denominator of the Beta distribution is defined as

$$\begin{aligned} B(a,b) &= \int_0^1 \theta^{a-1}(1-\theta)^{b-1}\mathrm{d}\theta \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad \text{where } a, b > 0 \end{aligned}$$

where

$$\Gamma(n) = (n-1)! \qquad \text{where } n > 0$$

[3 marks]



(a) (b)

Figure 1