A10613W1

TRINITY TERM 2018

SECOND PUBLIC EXAMINATION

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (PAPER C24)

Honour School of Engineering Science

Thursday 07 June 2018 09:30 – 11:00

Answers to not more than **THREE** *questions should be submitted, and each question must be answered in a separate booklet.*

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Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*

1. (a) Define and explain the notions of an ω limit point, an α limit point, an ω limit set, and an α limit set.

[4 marks]

(b) (i) If the point x is a fixed point, what is its ω limit point $\omega(x)$?

(ii) If the point x is on a periodic orbit, what is its ω limit point $\omega(x)$?

(iii) Figure 1 shows a saddle point x_0 with a homoclinic orbit attached, and an unstable focus point x_1 inside the homoclinic orbit such that trajectories inside spiral out from the centre to the edge. Let y_0 be a point on the homoclinic orbit, y_1 a point on an orbit outside it, and y_2 a point on a spiral inside it.
Determine each of $\omega(x_0)$, $\omega(y_0)$, $\omega(x_1)$, $\omega(y_1)$, and $\omega(y_2)$.

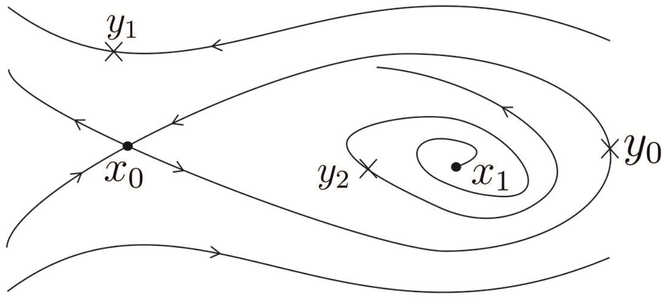

Figure 1

[5 marks]

(c) Consider the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \boldsymbol{x} \, .$$

(i) Determine the stable, unstable, and centre subspaces, and sketch the phase portrait.

(ii) Determine $\omega(\boldsymbol{x})$ for a point $\boldsymbol{x}$ lying on the $[\ 1 \quad 0 \quad 0\ ]$-axis, on the $[\ 0 \quad 1 \quad 0\ ]$-axis, and on the $[\ 0 \quad 0 \quad 1\ ]$-axis.

[7 marks]

2. (a) Consider a dynamical system of the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -3x^2 + 4x - 1. \end{aligned}$$

(i) Show that this system is a Hamiltonian system by giving an expression for the Hamiltonian function.

(ii) Compute all equilibria of this system and characterise the linear stability properties of the equilibrium points. Using the Hartman-Grobman theorem, what can you conclude for the stability properties of the equilibria for the nonlinear system?

(iii) Use Lyapunov's theorem to show that the equilibrium point $(1, 0)$ is stable.
Hint: Consider the Hamiltonian function as candidate for a Lyapunov function.

(iv) Can the equilibrium point be asymptotically stable? Explain your answer.

[10 marks]

(b) (i) Explain the relation between Hamiltonian and gradient systems.

(ii) Construct a gradient system that is related to the above Hamiltonian system.

(iii) Compute all equilibria of the gradient system and characterise the linear stability properties of the equilibrium points.

(iv) Does the gradient system have a periodic solution? Explain your answer.

[6 marks]

3. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \varepsilon(1 - y^2)\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

where $y(0) = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t}(0) = 0$.

(a) Using multiple scales, with $t_1 = t$ and $t_2 = \varepsilon t$, and a perturbation series for y of the form

$$y = y_0 + \varepsilon y_1 + \cdots \ ,$$

show that

$$y_0 = a_0(t_2)\mathrm{e}^{\mathrm{i}t_1} + b_0(t_2)\mathrm{e}^{-\mathrm{i}t_1} \ .$$

[6 marks]

(b) By considering secular terms in the solution, show that the equations governing the behaviour of a_0 and b_0 are

$$2\frac{\mathrm{d}a_0}{\mathrm{d}t_2} = a_0(1 - a_0 b_0)$$

$$2\frac{\mathrm{d}b_0}{\mathrm{d}t_2} = b_0(1 - a_0 b_0) \ ,$$

and give the initial conditions for a_0 and b_0.

[10 marks]

4. Consider the following dataset $\mathcal{D}$, tabulated as

x	$y(x)$
2	2
5	1 .

The goal is to model the function $y(x)$, where the independent variable x is positive and real. A model $\mathcal{M}$ is defined as

$$p(y \mid x, \theta, \mathcal{M}) = \begin{cases} x\, y\, \theta & 0 \le y < A \\ 0 & \text{otherwise} , \end{cases}$$

$$p(\theta \mid \mathcal{M}) = \begin{cases} \frac{1}{2} \sin\theta & 0 \le \theta < \pi \\ 0 & \text{otherwise} . \end{cases}$$

Observations y are identically and independently distributed given x, θ and $\mathcal{M}$.

(a) Determine A.

[3 marks]

(b) Determine the model evidence for $\mathcal{M}$ given $\mathcal{D}$.

[8 marks]

(c) Describe the model by interpreting $p(y \mid x, \theta, \mathcal{M})$ and $p(\theta \mid \mathcal{M})$. Characterise the functions $y(x)$ that are probable under this model.

[5 marks]