Paper: C24	Setter's name: Sina Ober-Blobaum	Page 1/4
Lecture title: Dynamical Systems		
Question no: **1**		

11

C 24 Dynamical Systems

Answers to Examination Question 2018

Question 1

1) a) • A point $p \in D$ is called an ω limit point of the trajectory $\Gamma c \phi(t,x)$ if there exists a sequence of times $\{t_i\}$, $t_i \to \infty$ such that

$$\lim_{i \to \infty} \phi(t_i, x) = p$$

This point is called $\omega(x)$.

- α limit points are defined in the same way, but with $t \to -\infty$. This point is called $\alpha(x)$
- $\alpha(\Gamma)$ and $\omega(\Gamma)$ are called the α-limit set and ω-limit set respectively, and they are the set of all α limit points and ω limit points for the trajectory Γ, i.e. the set of points from which the trajectory could have started and the set of points at which it ends.

b) (i) $\omega(x) = x$

(ii) $\omega(x)$ is the periodic orbit

(iii) $\omega(x_0) = x_0$, $\omega(y_0) = x_0$

$\omega(x_1) = x_1$, $\omega(y_1) = \phi$

$\omega(y_2) = \{x_0\} \cup \{\text{homoclinic orbit}\}$

Paper: C24	Setter's name: Sina Ober-Blobaum	Page 2/4
Lecture title: Dynamical Systems		
Question no: **1**		

To be handwritten. Please include question sections (e.g. a, b, c) and the marks attributed to each section.

2

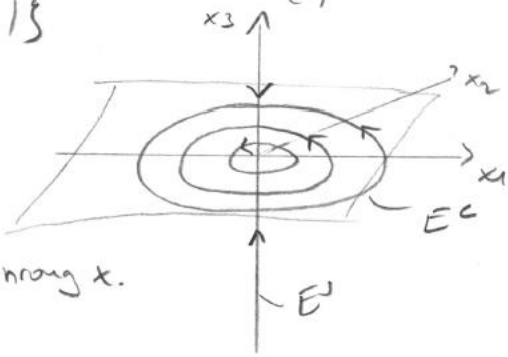
1) c) (i) eigenvalues $\lambda^c_{1,2} = \pm j$, eigenvectors $v^c = \begin{pmatrix} 1 \\ \pm j \\ 0 \end{pmatrix}$

$E^c = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

eigenvalue $\lambda^s_3 = -2$, eigenvector $v^s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$E^s = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$E^u = \emptyset$



(ii) - for x on the $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$-axis:
$w(x)$ is the periodic orbit in the x_1-x_2-plane going through x.

- for x on the $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$-axis:
$w(x)$ is the periodic orbit in the x_1-x_2-plane going through x.

- for x on the $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$-axis: $w(x) = (0,0,0)$

Paper: C24	Setter's name: Sina Ober-Blobaum	Page 3/4
Lecture title: Dynamical Systems		
Question no: **2**		

To be handwritten. Please include question sections (e.g. a, b, c) and the marks attributed to each section.

[5]

Question 2

2) a) (i) With $H(x,y) = \frac{1}{2}y^2 + x^3 - 2x^2 + x$

we obtain $\dot{x} = \frac{\partial H}{\partial y} = y$

$$\dot{y} = -\frac{\partial H}{\partial x} = -3x^2 + 4x - 1$$

(ii) equilibria $(1,0)$ and $(\frac{1}{3},0)$

Jacobian: $J = \begin{pmatrix} 0 & 1 \\ -6x+4 & 0 \end{pmatrix}$

$J(1,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \Rightarrow$ eigenvalues $\lambda_{1,2} = \pm j\sqrt{2}$

$\Rightarrow$ linear centre (non-hyperbolic)

$J(\frac{1}{3},0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \Rightarrow$ eigenvalues $\lambda_{1,2} = \pm\sqrt{2}$

$\Rightarrow$ saddle (hyperbolic) unstable

by Hartman-Grobman: the saddle $(\frac{1}{3}, 0)$ is also a non-linear saddle but we do not know about $(1,0)$ (because non-hyperbolic)

(iii) A natural Lyapunov function candidate is the Hamiltonian function

$$V(x,y) = \frac{1}{2}y^2 + x^3 - 2x^2 + x$$

a) $V(1,0) = 0$ ✓

b) $\dot{V}(x,y) = \nabla V(x,y) \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 0$ ✓

(either by computing or arguing that $V = H$ is an invariant of the system)

(c) $(1,0)$ is minimum of V because

$\nabla V(x,y) = \begin{pmatrix} 3x^2 - 4x + 1 \\ y \end{pmatrix}$, $\nabla^2 V(x,y) = \begin{pmatrix} 6x-4 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \nabla V(1,0) = 0$, $\nabla^2 V(1,0) > 0$ positive definite

$\Rightarrow V(x,y) > 0$ in a neighborhood of $(1,0)$ ✓

$\Rightarrow (1,0)$ is stable

Paper: C24	Setter's name: Sina Ober-Blobaum	Page 4/4
Lecture title: Dynamical Systems		
Question no: **2**		

To be handwritten. Please include question sections (e.g. a, b, c) and the marks attributed to each section.

[4

2) a) (iv) NO, because trajectories of Hamiltonian systems always evolve along level sets of the Hamiltonian which are closed curves. or:
Since Hamiltonian systems are conservative they cannot have attracting equilibrium points.

b) (i) A Hamilton system
$$\dot{x} = f(x,y) = -\frac{\partial H}{\partial y}$$
$$\dot{y} = g(x,y) = \frac{\partial H}{\partial x}$$
is orthogonal to the gradient system
$$\dot{x} = g(x,y), \quad \dot{y} = -f(x,y)$$
in the sense that trajectories of one system are orthogonal to trajectories of the other system

(ii)
$$\dot{x} = -3x^2+4x-1$$
$$\dot{y} = -y$$

(iii) Hamiltonian and gradient system have the same equilibria, centres map to nodes, saddles to saddles.

$\Rightarrow$ equilibria $(1,0)$ and $(\frac{1}{3},0)$

Jacobian $J(x,y) = \begin{pmatrix} -6x+4 & 0 \\ 0 & -1 \end{pmatrix}$

$J(1,0) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$ ev $\lambda_1 = -2, \lambda_1 = -1 \Rightarrow$ stable focus

$J(\frac{1}{3},0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$ ev $\lambda_1 = 2, \lambda_2 = -1 \Rightarrow$ unstable saddle

(iv) Gradient system cannot have periodic solutions
[Proof from lecture (not mandatory)]:
$\dot{x} = -\nabla V$; $\dot{V} = \nabla V \cdot \dot{x} = -\nabla V \cdot \nabla V = -|\dot{x}|^2$
if $x(t)$ is closed orbit of period T, then
$0 = V(x(t)) - V(x(0)) = -\int_0^T |\dot{x}|^2 \, dt \neq 0$ ↯

Paper: C24	Setter's name: **Stephen Payne**	Page 1 / 1
Lecture title: **Perturbation Methods**		
Question no: **3**		

a. Using multiple scales, with $t_1 = t$ and $t_2 = \varepsilon t$ yields:

$$\frac{d}{dt} \rightarrow \frac{dt_1}{dt}\frac{\partial}{\partial t_1} + \frac{dt_2}{dt}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}$$

Now inserting a perturbation series of the form:

$$y = y_0 + \varepsilon y_1 + \cdots$$

gives:

$$\left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}\right)\left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}\right)(y_0 + \varepsilon y_1 + \cdots) - \varepsilon(1 - (y_0 + \varepsilon y_1 + \cdots)^2)\left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}\right)(y_0 + \varepsilon y_1 + \cdots) + (y_0 + \varepsilon y_1 + \cdots) = 0$$

Collecting terms in ε^0 gives:

$$\frac{\partial^2 y_0}{\partial t_1^2} + y_0 = 0$$

hence the solution:

$$y_0 = a_0(t_2)e^{it_1} + b_0(t_2)e^{-it_1}$$

b. Collecting terms in ε^1 gives:

$$\frac{\partial^2 y_1}{\partial t_1^2} + y_1 = (1 - y_0^2)\frac{\partial y_0}{\partial t_1} - 2\frac{\partial^2 y_0}{\partial t_1 \partial t_2}$$

Substituting the answer from part (a) into the RHS then gives:

$$\frac{\partial^2 y_1}{\partial t_1^2} + y_1 = e^{it_1}\left[ia_0(1 - 2a_0b_0) + ia_0^2b_0 - 2i\frac{da_0}{dt_2}\right] + e^{-it_1}\left[-ib_0(1 - 2a_0b_0) - ia_0b_0^2 + 2i\frac{db_0}{dt_2}\right] + \cdots$$

The secular terms, i.e. the square brackets on the RHS, must be zero for the solution to be bounded:

$$2\frac{da_0}{dt_2} = a_0(1 - a_0b_0)$$
$$2\frac{db_0}{dt_2} = b_0(1 - a_0b_0)$$

The initial conditions come from the solution to part (a) and $y(0) = 1$ and $\frac{dy}{dt}(0) = 0$. Hence:

$$a_0(0) + b_0(0) = 1$$
$$ia_0(0) - ib_0(0) = 0$$

and thus: $a_0(0) = b_0(0) = 1/2$. Note that the question does not ask for a solution.

Paper: C24	Setter's name: Michael Osborne	Page 1/3
Lecture title: Advanced Probability		
Question no: **4**		

To be handwritten. Please include question sections (e.g. a, b, c) and the marks attributed to each section.

Consider the following dataset $\mathcal{D}$, tabulated as

x	$y(x)$
2	2
5	1.

Our goal is to model the function $y(x)$, where the independent variable x is positive and real. Define a model $\mathcal{M}$ as

$$p(y \mid x, \theta, \mathcal{M}) = \begin{cases} x\, y\, \theta & 0 \leq y < A \\ 0 & \text{otherwise} \end{cases}$$

$$p(\theta \mid \mathcal{M}) = \begin{cases} \frac{1}{2}\sin(\theta) & 0 \leq \theta < \pi \\ 0 & \text{otherwise.} \end{cases}$$

Observations y are identically and independently distributed given x, θ and $\mathcal{M}$.

(a) Determine A.

[3 marks]

> A can be determined by noting that $p(y \mid x, \theta, \mathcal{M})$ must be normalised over y. That is,
>
> $$\begin{aligned} 1 &= \int p(y \mid x, \theta, \mathcal{M})\, \mathrm{d}y \\ &= \int_0^A x\, y\, \theta\, \mathrm{d}y \\ &= \left[\frac{1}{2} x\, y^2\, \theta\right]_0^A \\ &= \frac{1}{2} x\, A^2\, \theta \\ A &= \sqrt{\frac{2}{x\,\theta}}. \end{aligned}$$
>
> Only the positive root is consistent with the question.

(b) Determine the model evidence for $\mathcal{M}$ given $\mathcal{D}$.

[8 marks]

Paper: C24	Setter's name: Michael Osborne	Page 2/3
Lecture title: Advanced Probability		
Question no: **4**		

To be handwritten. Please include question sections (e.g. a, b, c) and the marks attributed to each section.

Draft Only – 12th January 2018

Given that the two observations are IID, the model evidence is

$$p(\mathcal{D} \mid \mathcal{M}) = \int p(y=2 \mid x=2, \theta, \mathcal{M})\, p(y=1 \mid x=5, \theta, \mathcal{M})\, p(\theta \mid \mathcal{M})\, \mathrm{d}\theta$$
$$= \int_0^L (2\times 2)\,\theta\,(5\times 1)\,\theta\,\frac{1}{2}\sin(\theta)\,\mathrm{d}\theta.$$

L is determined by recognising that $p(y \mid x, \theta, \mathcal{M}) = 0$ if

$$y \geq A$$
$$y \geq \sqrt{\frac{2}{x\theta}}$$
$$\frac{1}{2}y^2 x \geq \frac{1}{\theta}$$
$$\frac{2}{y^2 x} \leq \theta.$$

There are two such terms in the integrand ($p(y=2 \mid x=2, \theta, \mathcal{M})$ and $p(y=1 \mid x=5, \theta, \mathcal{M})$): The first term is zero for $\theta \geq \frac{2}{y^2 x} = \frac{2}{2^2\times 2} = \frac{1}{4}$. The second term is zero for $\theta \geq \frac{2}{y^2 x} = \frac{2}{1^2\times 5} = \frac{2}{5}$. $p(\theta \mid \mathcal{M})$ is zero for $\theta \geq \pi$. L is given by the lowest of the three limits, that is, $L = \frac{1}{4}$.
Returning to the model evidence,

$$p(\mathcal{D} \mid \mathcal{M}) = 10\int_0^{\frac{1}{4}} \theta^2 \sin(\theta)\,\mathrm{d}\theta.$$

Recalling integration by parts (which we'll use twice):

$$(u\,v)' = u'\,v + v'\,u$$
$$u\,v - \int v'\,u\,\mathrm{d} = \int u'\,v.$$

Hence

$$p(\mathcal{D} \mid \mathcal{M}) = 10\left[\theta^2(-\cos(\theta))\right]_0^{\frac{1}{4}} - 10\int_0^{\frac{1}{4}} 2\theta(-\cos(\theta))\,\mathrm{d}\theta$$
$$= 10\left[\theta^2(-\cos(\theta))\right]_0^{\frac{1}{4}} + 20\int_0^{\frac{1}{4}} \theta\cos(\theta)\,\mathrm{d}\theta$$
$$= -\frac{5}{8}\cos\left(\frac{1}{4}\right) + 20\left(\left[\theta\sin(\theta)\right]_0^{\frac{1}{4}} - \int_0^{\frac{1}{4}}\sin(\theta)\,\mathrm{d}\theta\right)$$
$$= -\frac{5}{8}\cos\left(\frac{1}{4}\right) + 5\sin\left(\frac{1}{4}\right) - 20\left[-\cos(\theta)\right]_0^{\frac{1}{4}}$$
$$= -\frac{5}{8}\cos\left(\frac{1}{4}\right) + 5\sin\left(\frac{1}{4}\right) + 20\cos\left(\frac{1}{4}\right) - 20$$
$$= \frac{155}{8}\cos\left(\frac{1}{4}\right) + 5\sin\left(\frac{1}{4}\right) - 20 \simeq 0.00970.$$

:) Describe the model by interpreting $p(y \mid x, \theta, \mathcal{M})$ and $p(\theta \mid \mathcal{M})$. Characterise the functions

Paper: C24	Setter's name: Michael Osborne	Page 3/3
Lecture title: Advanced Probability		
Question no: **4**		

To be handwritten. Please include question sections (e.g. a, b, c) and the marks attributed to each section.

Draft Only – 12th January 2018

$y(x)$ that are probable under this model.

[5 marks]

- $p(y \mid x, \theta, \mathcal{M})$ is the likelihood (connecting the parameter θ to data) and $p(\theta \mid \mathcal{M})$ is the prior for the parameter.
- $\mathcal{M}$ is a parametric model: it has only the single parameter θ. It is hence likely to be far less flexible than a non-parametric model, like a Gaussian process.
- $p(\theta \mid \mathcal{M})$ is a fairly typical unimodal, bounded-support, prior, with a mean and mode of $\frac{\pi}{2}$.
- $p(y \mid x, \theta, \mathcal{M})$ is a slightly unusual likelihood. For a given x and θ, the probability of y is linearly related to the value of y: the most probable value is just less than A.
- According to the likelihood, $0 \leq y(x) < \sqrt{\frac{2}{x\theta}}$. As such, the maximum possible value of $y(x)$ *decreases* with increasing x and θ. However, within the envelope of possible values for $y(x)$, the weighting towards higher values linearly *increases* with increasing x and θ.