A10613W1

TRINITY TERM 2019

SECOND PUBLIC EXAMINATION

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (Paper C24)

Honour School of Engineering Science

Monday 17 June 2019 09:30 – 11:00

Answers to not more than **THREE** *questions should be submitted, and each question must be answered in a separate booklet.*

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Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*

1. A second order dynamical system is defined by the equations

$$\dot{x}_1 = -2x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2 + x_1 x_2 \ .$$

(a) Find the equilibrium points of the system, determine the linearization about each equilibrium point and classify its stability.

[6 marks]

(b) Sketch the phase portrait of the system, showing clearly the behaviour of the system close to each equilibrium point and far from the equilibrium points.

[4 marks]

(c) (i) Show that each of the following three sets

$$\{(x_1, x_2) \ : \ x_1 \le 0, \ x_2 \ge 0, -x_1 + x_2 \le a\}$$
$$\{(x_1, x_2) \ : \ x_1 \le 0, \ x_2 \le 0, -x_1 - x_2 \le a\}$$
$$\{(x_1, x_2) \ : \ x_1 \ge 0, \ x_2 \le 0, \quad x_1 - x_2 \le a\}$$

is positively invariant for any constant $a > 0$.

[4 marks]

(ii) Hence or otherwise show that the origin is the ω limit point of any solution trajectory passing through a point in the second, third or fourth quadrant.

[2 marks]

2. (a) The condition

$$\oint_{\Gamma} \Big(f_2(x_1, x_2)\, \mathrm{d}x_1 - f_1(x_1, x_2)\, \mathrm{d}x_2 \Big) = 0$$

must be satisfied on any periodic orbit Γ of the second order system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \ . \end{aligned}$$

Use this condition to show that the system cannot have a periodic orbit lying entirely in a region D if

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$$

is satisfied everywhere in D.

[4 marks]

(b) Consider the system defined by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^3 + x_1 x_2^2 \\ \dot{x}_2 &= -x_2 + x_2^3 + x_1^2 x_2 \ . \end{aligned}$$

(i) Show that this system cannot have a limit cycle that is contained entirely in the set

$$D = \{(x_1, x_2) : x_1^2 + x_2^2 < \tfrac{1}{2}\} \ .$$

[2 marks]

(ii) Show that D is positively invariant, and explain what this implies about the existence of a limit cycle intersecting D.

[6 marks]

(iii) Explain why this system cannot have a limit cycle.

[4 marks]

3. Consider the following differential equation

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = a - \frac{\mathrm{d}y}{\mathrm{d}x}$$

in the range $0 \leq x \leq 1$, where $y(0) = 0$ and $y(1) = 1$, and a is a positive constant with $a < 1$.

(a) Explain why we expect a boundary layer to feature in the solution to this differential equation.

[2 marks]

(b) Using asymptotic matching and a stretching co-ordinate for the boundary layer of the form

$$\overline{x} = \frac{x}{\varepsilon^{\alpha}} \ ,$$

show that the approximate solution is

$$y = ax + (1 - a)(1 - e^{-x/\varepsilon}) \ .$$

[12 marks]

(c) Sketch the resulting solution and indicate the position of the boundary layer.

[2 marks]

4. Movies are rated by a user on a three-star scale. That is, defining two such ratings as X and Y, we have $X, Y \in \{1, 2, 3\}$. A-priori, no rating is more probable than any other. Assume that

$$P(X = 1, Y = 1) = P(X = 1, Y = 2) = P(X = 2, Y = 1) = P(X = 2, Y = 2) = q,$$

where q is a parameter. Assume throughout that $0 \log 0 = 0$.

(a) Calculate the entropy of X.

[3 marks]

(b) (i) Given q, calculate $P(X = x, Y = y)$.

(ii) State the range of possible values for q.

[5 marks]

(c) Assume that q takes its largest possible value.

(i) Write out $P(X = x, Y = y)$.

(ii) Find the mutual information between X and Y.

(iii) Comment on your answer.

[8 marks]