C24 2019 Questions 1 & 2: Dynamical Systems (MRC)

1. A second order dynamical system is defined by the equations

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_1x_2\\ \dot{x}_2 &= -x_2 + x_1x_2\end{aligned}$$

(a) Find the equilibrium points of the system, determine the linearization about each equilibrium point and classify its stability.

[6 marks]

(b) Sketch the phase portrait of the system, showing clearly the behaviour of the system close to each equilibrium point and far from the equilibrium points.

[4 marks]

(c) (i) Show that each of the following three sets

$$\begin{aligned}&\{(x_1, x_2) \,:\, x_1 \leq 0,\ x_2 \geq 0, -x_1 + x_2 \leq a\}\\ &\{(x_1, x_2) \,:\, x_1 \leq 0,\ x_2 \leq 0, -x_1 - x_2 \leq a\}\\ &\{(x_1, x_2) \,:\, x_1 \geq 0,\ x_2 \leq 0, \quad x_1 - x_2 \leq a\}\end{aligned}$$

is positively invariant for any constant $a > 0$.

[4 marks]

(ii) Hence or otherwise show that the origin is the ω limit point of any solution trajectory passing through a point in the second, third or fourth quadrant.

[2 marks]

2. (a) The condition

$$\oint_\Gamma f_2(x_1, x_2)\,\mathrm{d}x_1 - f_1(x_1, x_2)\,\mathrm{d}x_2 = 0$$

must be satisfied on any periodic orbit Γ of the second order system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

Use this condition to show that the system cannot have a periodic orbit lying entirely in a region D if

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$$

is satisfied everywhere in D.

[4 marks]

(b) Consider the system defined by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^3 + x_1 x_2^2 \\ \dot{x}_2 &= -x_2 + x_2^3 + x_1^2 x_2 \end{aligned}$$

(i) Show that this system cannot have a limit cycle that is contained entirely in the set

$$D = \{(x_1, x_2) : x_1^2 + x_2^2 < \tfrac{1}{2}\}$$

[2 marks]

(ii) Show that D is positively invariant, and explain what this implies about the existence of a limit cycle intersecting D.

[6 marks]

(iii) Explain why this system cannot have a limit cycle.

[4 marks]

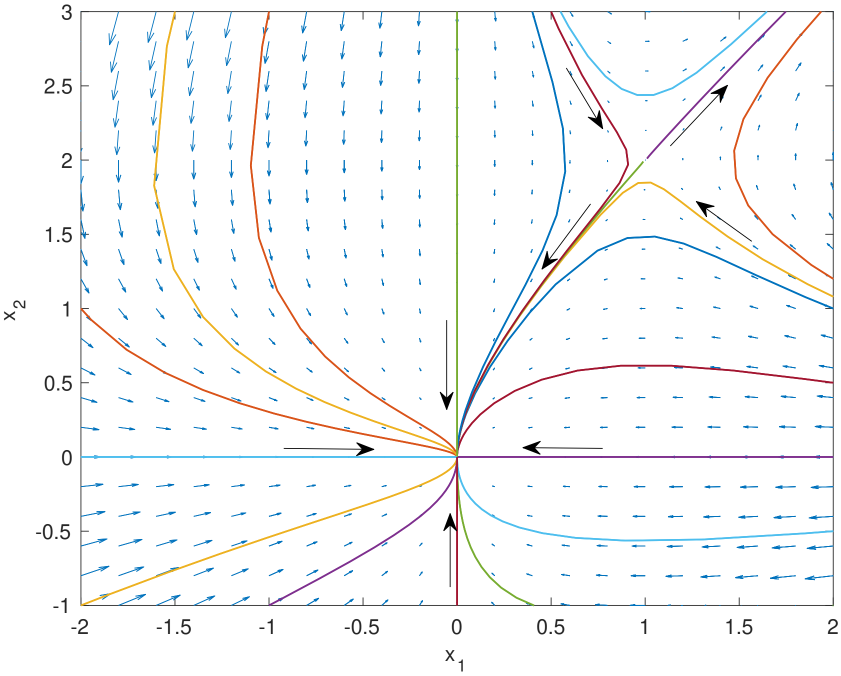
C24 2019 Solutions to questions 1 & 2 (MRC)

1. (a) $\dot{x}_1 = 0 \implies x_1 = 0$ or $x_2 = 2$,
$\dot{x}_2 = 0 \implies x_2 = 0$ or $x_1 = 1$, so $(0,0)$ and $(1,2)$ are the only equilibrium points.

(i) Linearize about $(0,0)$: $\dot{x} = -\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x$, eigenvalues/vectors: $\left(-2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
so this is a stable node.

(ii) Linearizing about $(1,2)$: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} x$, eigenvalues/vectors: $\left(\sqrt{2}, \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\right), \left(-\sqrt{2}, \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}\right)$
so this is an unstable saddle.

(b) Phase portrait:



Note the positions of the two equilibria, their (locally) invariant directions, the saddle separatrices and the trajectories lying on the x_1- and x_2-axis.

(c) (i) If $x_1 \le 0$ and $x_2 \ge 0$, then $-\dot{x}_1 + \dot{x}_2 = 2x_1 - x_2 \le 0$.
If $x_1 \le 0$ and $x_2 \le 0$, then $-\dot{x}_1 - \dot{x}_2 = 2x_1 + x_2 - 2x_1x_2 \le 0$.
If $x_1 \ge 0$ and $x_2 \le 0$, then $\dot{x}_1 - \dot{x}_2 = -2x_1 + x_2 \le 0$.
These inequalities and the property that $\dot{x}_1 = 0$ when $x_1 = 0$ and $\dot{x}_2 = 0$ when $x_2 = 0$ imply that none of the solution trajectories on the boundaries of the intersections of the 2nd, 3rd and 4th quadrants with the diamond defined by $\{(x_1, x_2) : |x_1 \pm x_2| \le a\}$ can escape these sets, and therefore these three sets are positively invariant.

(ii) From the Poincaré-Bendixson Theorem we can conclude that all solution trajectories passing through points in the 2nd, 3rd and 4th quadrants converge to the origin, since the origin is the only equilibrium point in each of these sets, and since a is arbitrary.

2. (a) Any solution trajectory satisfies $\dot{x}_1/\dot{x}_2 = \mathrm{d}x_1/\mathrm{d}x_2 = f_1/f_2$ and hence $f_2\,\mathrm{d}x_1 - f_1\,\mathrm{d}x_2 = 0$. Suppose that the system has a closed orbit Γ, then $\oint_\Gamma f_2\,\mathrm{d}x_1 - f_1\,\mathrm{d}x_2 = 0$, and since this is the integral of the vector $[-f_2 \;\; f_1]^\top$ around the closed contour Γ, we can apply Stokes' Theorem to get

$$\int_S \Big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0$$

where S is the region of plane enclosed by Γ. But this condition cannot be satisfied if $\partial f_1/\partial x_1 + \partial f_2/\partial x_2 < 0$ (or > 0) everywhere in any set D containing S, implying that the system cannot have a closed orbit Γ lying entirely in D. [This is the Bendixson criterion.]

(b) (i) With $f_1 = -x_1 + x_1^3 + x_1x_2^2$ and $f_2 = -x_2 + x_2^3 + x_1^2x_2$ we get

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -2 + 4x_1^2 + 4x_2^2 < 0 \quad \text{for all } (x_1, x_2) \in D$$

where D is the open disc with centre $(0,0)$ and radius $\frac{1}{\sqrt{2}}$, implying that D cannot contain any periodic orbits.

(ii) To demonstrate postive invariance of D, consider the function $V(x_1, x_2) = x_1^2 + x_2^2$. The time-derivative of V along any solution trajectory is

$$\begin{aligned}\tfrac{1}{2}\dot{V} &= -x_1^2 + x_1^3 + x_1^2x_2^2 - x_2^2 + x_2^4 + x_1^2x_2^2 \\ &= (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \\ &= V(V-1)\end{aligned}$$

and hence $\dot{V} < 0$ for all $V \in (0,1)$. Since D is equivalently defined as the set on which $V < 1/2$, this implies that D is a positively invariant set.

The significance of this for limit cycles intersecting D is that D cannot contain any part of a limit cycle, since no orbit passing through D can leave D and from (i) no closed orbit is contained entirely within D.

(iii) Using the partial derivatives found in (b)(i) we find that no limit cycle is contained entirely within the set $E = \{(x_1, x_2) : x_1^2 + x_2^2 > \frac{1}{2}\}$, since $\partial f_1/\partial x_1 + \partial f_2/\partial x_2 > 0$ for all $(x_1, x_2) \in E$.

Combining result this with conclusion from (b)(ii), the only remaining possibility for the location of a limit cycle is the circle with centre $(0,0)$ and radius $\frac{1}{\sqrt{2}}$. However, this circle cannot be a closed orbit of the system because V is constant ($V = \frac{1}{2}$) and $\dot{V}$ is non-zero ($\dot{V} = -\frac{1}{2}$) everywhere on it, and therefore the system cannot have a limit cycle.