A10613W1

TRINITY TERM 2021

SECOND PUBLIC EXAMINATION

Honour School of Engineering Science

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (Paper C24)

Thursday 10 June 2021 Opening Time: 09:30am UK Time

Mode of Completion: Handwritten

Answers to not more than **THREE** *questions should be submitted.*

Start each question on a new page. You must upload your answer files into the submission box for the relevant question. Questions with sub-sections (e.g. Q1a & Q1b) should all be included in the answer file for the corresponding overall question number and uploaded into the submission box for the relevant question.

You have 1 hour and 30 minutes writing time to complete the paper and up to 30 minutes technical time to upload your answer files.

Note that:

- *The approximate allocation of marks is given in the margin.*
- *You are permitted to use the following material(s):*
- *Engineering Tables & Data (HLT) - Candidate to provide.*

1. (a) State the Hartman-Grobman theorem and explain what it means.

[4 marks]

(b) Consider a dynamical system of the form

$$\ddot{x} - \cos x = 0 \ .$$

(i) Transform this second-order system into a system of two first-order equations.

(ii) Show that the first-order system from part (i) is a Hamiltonian system by giving an expression for the Hamiltonian function.

(iii) Characterise the linear stability properties of all equilibria of this system, and sketch its phase portrait. Using the Hartman–Grobman theorem, what can you conclude for the stability properties of the equilibria for the nonlinear system?

(iv) Use Lyapunov's theorem to show that the equilibrium point $(\pi/2, 0)$ is stable. Hint: Consider the Hamiltonian function as candidate for a Lyapunov function.

[8 marks]

(c) Consider the modified dynamical system of the form

$$\ddot{x} - \cos x = \varepsilon \dot{x}$$

with $\varepsilon < 0$. Use LaSalle's invariance principle to show that the equilibrium $(\pi/2, 0)$ of this system is asymptotically stable.

[4 marks]

2. (a) State the conditions that have to be satisfied for a system undergoing a Hopf bifurcation at $\mu = \mu_0$. Describe what happens for a supercritical and a subcritical Hopf bifurcation at values ranging from $\mu < \mu_0$ to $\mu > \mu_0$, and sketch the system's phase portraits when $\mu < \mu_0$ and $\mu > \mu_0$.

[5 marks]

(b) Consider a dynamical system of the form

$$\dot{x}_1 = x_1\left(-\mu + x_1^2 + x_2^2\right) + x_2$$
$$\dot{x}_2 = x_2\left(-\mu + x_1^2 + x_2^2\right) - x_1 \quad ,$$

where μ is a real-valued parameter. Characterise the linear stability properties of the equilibrium $(0, 0)$ for positive, negative, and null values of μ.

[4 marks]

(c) (i) Show that a two-dimensional nonlinear system with states x_1 and x_2 can be written in polar coordinates using

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$$
$$\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2} \quad .$$

(ii) Transform the system from part (b) into polar coordinates (r, θ).

(iii) Consider the system in polar coordinates for $\mu > 0$. Show that this system has a limit cycle, find it and determine its stability.

(iv) Conclude from your results what kind of bifurcation the system undergoes at $\mu = 0$.

[7 marks]

3. Consider the following differential equation:

$$\varepsilon^2 \frac{d^2y}{dx^2} - q(x)y = 0$$

in the range $0 < x < 1$, where $y(0) = a$ and $y(1) = b$.

(a) Explain the idea behind the WKB method, where we assume a solution of the form:

$$y = e^{\theta(x)/\varepsilon^\alpha} \left[y_0(x) + \varepsilon^\alpha y_1(x) + ...\right]$$

where the symbols have their usual meaning.

[2 marks]

(b) Given that $q(x) = -e^{2x}$ and using the following results:

$$(\theta')^2 - q(x) = 0$$

$$\theta'' y_0 + 2\theta' y_0' = 0$$

derive the general WKB solution for the given differential equation.

[9 marks]

(c) Now consider the differential equation:

$$\frac{d^2y}{dx^2} + \lambda^2 e^{2x} y = 0$$

in the range $0 < x < 1$, where $y(0) = 0$ and $y(1) = 0$. To produce a non-zero solution, show that the following equation for λ must be satisfied:

$$\lambda = \frac{n\pi}{e - 1}$$

[5 marks]

4. (a) You can use any results from the lecture and example sheet to address the following points:

(i) Consider n functions $\phi_i : \mathbb{R} \to \mathbb{R}$ ($i \in \{1, ..., n\}$). Assume $W \sim \mathcal{N}(0, C)$ where $C \in \mathbb{R}^{n \times n}$ is a covariance matrix and define $F_x := f(x; W)$ where $f(x; W) = \sum_{i=1}^{n} W_i \phi_i(x)$. What are the mean and variance functions $m(x) = \mathbb{E}[F_x] = \langle F_x \rangle$, $v(x) = \text{var}[F_x]$? (Formula suffices, no derivation necessary).

(ii) Prove or refute the statement: "$k : (x, \xi) \mapsto x^2 \xi^2$ is a valid covariance function on $\mathbb{R}^2$."

(iii) Prove or refute the statement: "$k : \begin{cases} \mathbb{R}_+ \to \mathbb{R} \\ (x, \xi) \mapsto x \xi^2 \end{cases}$ is a valid covariance function on the set of positive numbers $\mathbb{R}_+$."

[6 marks]

(b) Let X be a discrete random variable given by a Poisson distribution with probability mass function

$$p(X = x; \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!} \quad .$$

By appealing to the (e.g, canonical) form of the exponential family of distributions, show that the distribution of X belongs to the exponential family of distributions.

[4 marks]

(c) Consider a one-dimensional Bayesian linear regression problem where we have one noisy function value observation $y = 1$ of a linear function $f : x \mapsto wx$ at input $x = 1$. Here, we assume $y = f(x; w) + v$ where the (unknown) noise perturbation v was drawn from a Laplace distribution with density $p(v) = \frac{1}{2} \exp(-|v|)$. Thus, $p(y|x, w) = \frac{1}{2} \exp(-|y - f(x; w)|)$. We assume slope parameter w is *positive* but otherwise uncertain. We model our Bayesian belief about w by exponentially distributed random variable W with density $p(W = w) = \exp(-w)$.
Derive:

(i) A maximum likelihood estimate $w_{ml} \in \arg\max_w p(y|W = w, x)$ of the slope parameter w.

(ii) A maximum a posteriori estimate $w_{map} \in \arg\max_w p(W = w|y, x)$ of the slope parameter w.

Make sure to not just provide the numerical values but also to show how you derived them.

[6 marks]