C 24: Dynamical Systems

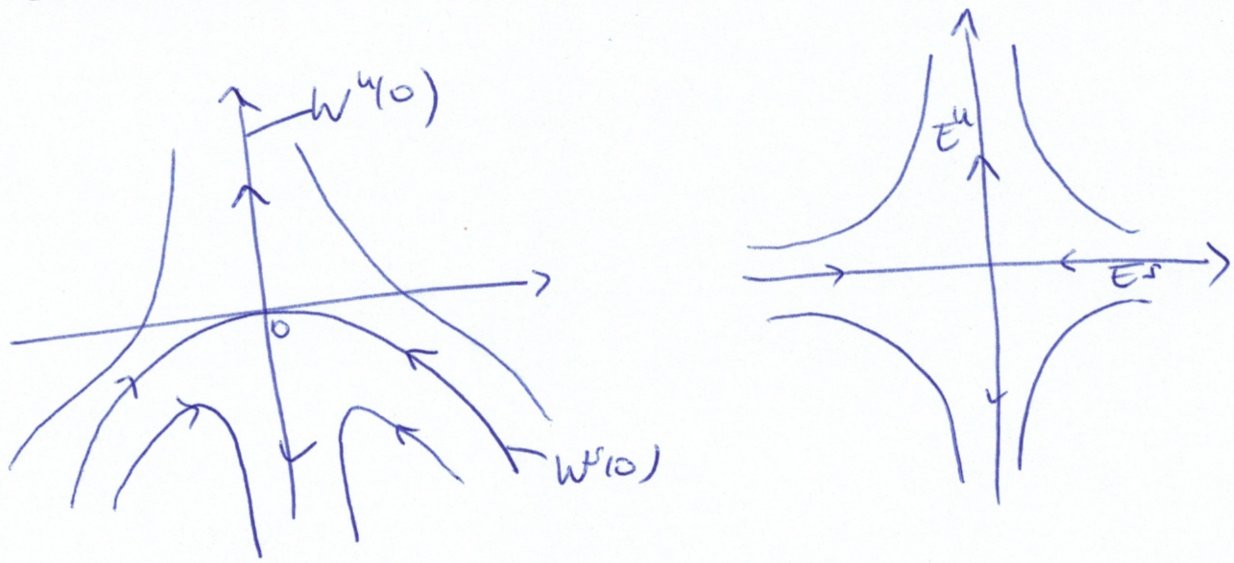
Answers to Examination Questions 2020

Question 1

1. (a) In the case of hyperbolic equilibria, there exists a bi-continous function H of a region containing the equilibrium (of the phase-space of the nonlinear system) to a region containing the origin (of the phase-space of the linearization) such that trajectories are mapped exactly, and parametrisation of time is preserved

Meaning:
- Near the equilibrium, the stable linear subspace is mapped to a stable manifold in a region surrounding the equilibrium point
- Near the equilibrium, the unstable linear subspace is mapped to an unstable manifold in a region surrounding the equilibrium point
- Nothing is said about centers.

Example:



(picture not compulsive)

1. (b) $\ddot{x} - \cos x = 0$

(i) $x_1 = x,\ x_2 = \dot{x} \Rightarrow \dot{x}_1 = x_2,\ \dot{x}_2 = \cos x_1$

(ii) $H(x_1, x_2) = \frac{1}{2} x_2^2 - \sin x_1 \Rightarrow \dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2$

$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = \cos x_1$

(iii) equilibria: $x^* = \left(\frac{\pi}{2} + k \cdot \pi, 0\right) \quad k = \dots, -2, -1, 0, 1, 2, \dots$

Jacobian $J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -\sin x_1 & 0 \end{pmatrix}$

Distinguish between $\tilde{x} = \left(\frac{\pi}{2} + k \cdot \pi, 0\right),\ k = \dots -2, 0, 2, 4 \dots$

and $x^* = \left(\frac{\pi}{2} + k \cdot \pi, 0\right),\ k = \dots -3, -1, 1, 3, \dots$

$\Rightarrow J(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad J(x^*) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

eigenvalues: of $J(\tilde{x})$: $\lambda_{1,2} = \pm i$ (non-hyperbolic)

of $J(x^*)$: $\lambda_{1,2} = \pm 1$ (hyperbolic)

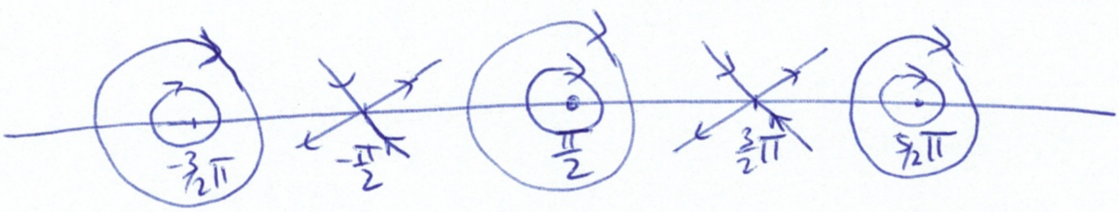
$\Rightarrow \tilde{x}$ is linear centre, x^* is saddle (unstable)

eigenvectors: $\tilde{x}$: $\lambda_1 = i,\ v_1 = \begin{pmatrix} -1 \\ i \end{pmatrix}$

$\lambda_2 = -i,\ v_2 = \begin{pmatrix} -1 \\ -i \end{pmatrix}$

x^*: $\lambda_1 = -1 \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow E^s = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$\lambda_2 = +1 \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow E^u = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$



by Hartmann-Grobman: x^* also nonlinear saddle, but we do not know about $\tilde{x}$ (because non-hyperbolic)

(iv) A natural Lyapunov function candidate is the Hamiltonian function $\tilde{V}(x_1,x_2) = \frac{1}{2}x_2^2 - \sin x_1$

However $\tilde{V}(\frac{\pi}{2},0) = -1 \Rightarrow$ choose $V(x_1 x_2) = \frac{x_2^2}{2} - \sin x_1 + 1$

$\Rightarrow V(\frac{\pi}{2},0) = 0$ and $V(x_1,x_2) > 0$ for $(x_1,x_2) \neq (0,0)$

$\Rightarrow$ condition (i) is satisfied

Compute $\dot{V}(x_1,x_2) = \nabla V(x_1,x_2) \cdot f(x) = x_2\dot{x}_2 - \cos x_1 \dot{x}_1$

$= x_2(\cos x_1) - \cos x_1 \dot{x}_1 = 0$

$\Rightarrow$ condition (ii) is satisfied $\Rightarrow (\frac{\pi}{2},0)$ is stable.

1. (c) first order system: $\dot{x}_1 = x_2$, $\dot{x}_2 = \cos x_1 + \varepsilon x_2$

Consider $V(x_1,x_2) = \frac{1}{2}x_2^2 - \sin x_1 + 1$

$\Rightarrow \dot{V}(x_1 x_2) = \varepsilon x_2^2 \leq 0$ $(\varepsilon < 0)$ is negative everywhere except on the line $x_2 = 0$ where $\dot{V} = 0$

$\Rightarrow$ set E consists of all points with $x_2 = 0$, but only the points $x_1 = \frac{\pi}{2} + k\cdot\pi$, $k \in \mathbb{Z}$ are invariant.

The level set $V = c$ with $0 < c < \pi$ is the boundary of a positively invariant region and that region contains only the equilibrium point $(\frac{\pi}{2},0)$

$\Rightarrow$ this defines the positively invariant set D.

$\Rightarrow E = \{x \in D$ such that $\dot{V}(x) = 0\} = \{x \in D, x_2 = 0\}$

$\Rightarrow M = \{(\frac{\pi}{2},0)\} \xrightarrow{\text{LaSalle}}$ all trajectories starting at $x \in D$ tend to M as $t \to \infty$

$\Rightarrow (\frac{\pi}{2},0)$ is asymptotically stable.

Question 2

2)(a) 1. For $\mu < \mu_0 + \varepsilon$ and $\mu > \mu_0 - \varepsilon$ for $\varepsilon > 0$

$$\lambda_{1,2}(\mu) = \alpha(\mu) \pm j\omega(\mu)$$

2. $\alpha(\mu_0) = 0$
3. $\alpha(\mu) < 0$ for $\mu < \mu_0$
4. $\alpha(\mu) > 0$ for $\mu > \mu_0$

Supercritical Hopf bifurcation:

For $\mu < \mu_0$ we have a stable spiral fixed point which for $\mu > \mu_0$ becomes an unstable spiral. The unstable spiral is bounded by a stable limit cycle which expands with increasing μ

$\mu < \mu_0$

$\mu > \mu_0$

Subcritical Hopf bifurcation:

For $\mu < \mu_0$ a stable spiral is surrounded by an unstable limit cycle. As μ increases, the unstable limit cycle becomes smaller and at $\mu = \mu_0$ the cycle collapses on a fixed point which for $\mu > \mu_0$ behaves as an unstable spiral

$\mu < \mu_0$

$\mu > \mu_0$

(2) b) $\dot{x}_1 = x_1(-\mu + x_1^2 + x_2^2) + x_2$

$\dot{x}_2 = x_2(-\mu + x_1^2 + x_2^2) - x_1$

linearisation in (0,0): $J = \begin{pmatrix} -\mu & 1 \\ -1 & -\mu \end{pmatrix}$

eigenvalues $\lambda_{1,2} = -\mu \pm i$

$\mu > 0$: (0,0) is stable spiral
$\mu = 0$: (0,0) is centre
$\mu < 0$: (0,0) is unstable spiral

(2) c) (i) Since $r^2 = x_1^2 + x_2^2$ differentiating gives

$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2 \Rightarrow \dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$

Similarly, $\tan\theta = \frac{x_2}{x_1} \Rightarrow (1+\tan^2\theta)\dot{\theta} = \frac{-\dot{x}_1 x_2 + \dot{x}_2 x_1}{x_1^2}$

$\Rightarrow \dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$

(ii) polar coordinates: $\dot{r} = \dots = -\mu r + r^3$

$\dot{\theta} = \dots = -1$

(iii) $\dot{r} = -\mu r + r^3$, $\mu > 0$

$\dot{\theta} = -1$

$r = \sqrt{\mu}$: $\dot{r} = 0 \Rightarrow$ limit cycle

$\left.\begin{array}{ll} r < \sqrt{\mu} : & \dot{r} < 0 \\ r > \sqrt{\mu} : & \dot{r} > 0 \end{array}\right\}$ limit cycle is unstable

(iv) Since there is an unstable limit cycle for $\mu > \mu_0 = 0$ we have a subcritical Hopf bifurcation