A10613W1

TRINITY TERM 2022

SECOND PUBLIC EXAMINATION

Honour School of Engineering Science

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (Paper C24)

08 June 2022 9:30 a.m. – 11:00 a.m.

Answers to not more than **THREE** *questions should be submitted.*

Mode of Completion: Handwritten

You have 1 hour and 30 minutes writing time to complete the paper and up to 30 minutes technical time to upload your answer files.

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Note that:

- *The approximate allocation of marks is given in the margin.*
- *You are permitted to use the following material(s):*
- *Engineering Tables & Data (HLT) - Candidate to provide.*

1. Consider the initial-value problem,

$$c y \frac{\mathrm{d}y}{\mathrm{d}t} = 1 - t - y \quad \text{with} \quad y(0) = 0\,,$$

where c is a positive constant that can be either small or large.

(a) First, consider $c = \epsilon \ll 1$ (i.e., small c).

(i) Use a standard perturbation expansion, $y = y_0 + \epsilon y_1 + \ldots$, to show that there must be a boundary layer at $t = 0$. Derive the leading-order outer solution, $y_0(t)$.

Hint: $\int x(1-x)^{-1}\mathrm{d}x = -\ln|1-x| - x\,.$

(ii) Use a stretched coordinate $\bar{t} = t/\epsilon$ to derive *an implicit algebraic expression* for the leading-order inner solution, $\bar{y}_0(t)$, making sure to impose the initial condition.

(iii) Derive a composite solution by matching your outer and inner solutions. You can leave the result in terms of $\bar{y}_0$.

[8 marks]

(b) Now, consider $c^{-1} = \epsilon \ll 1$ (i.e., large c).

(i) Introduce a perturbation expansion of the form $y = \epsilon^{\alpha}(y_0 + \epsilon^{\gamma} y_1 + \ldots)$ and determine the value of α that provides a balance at leading order. Derive the leading-order solution $y_0(t)$.

(ii) Determine the value of γ that provides a balance at the next order and thus derive the associated ODE and initial condition for $y_1(t)$. Do not solve it.

(iii) Write an expression for the solution $y(t)$ valid to first order in ϵ. You can leave the result in terms of y_1.

[8 marks]

2. A second order system is defined by:

$$\dot{x} = -y - \mu\left(x^5/5 - x\right), \text{ and}$$
$$\dot{y} = x,$$

where μ is a constant.

(a) Find the fixed points of the system and use linearisation to classify them when $\mu < 0$ and when $\mu > 0$.

[4 marks]

(b) For $\mu < 0$, use the function $V(x, y) = x^2 + y^2$ to show that every trajectory of the system starting from an initial condition (x_0, y_0) satisfying

$$x_0^2 + y_0^2 < \sqrt{5}$$

has $(x, y) = (0, 0)$ as its ω-limit point.

[8 marks]

(c) For $\mu > 0$, the system is known to have a single limit cycle C enclosing $(x, y) = (0, 0)$. Explain why every trajectory starting from a non-zero initial condition (x_0, y_0) that belongs to the set enclosed by C must have C as its ω-limit set

[4 marks]

3. Cell growth in a biopharmaceutical manufacturing process is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x) - \frac{hx}{(x + a)},$$

where $x(t)$ is proportional to cell density, and h and a are constant parameters satisfying $h > 0$ and $0 < a < 1$.

(a) Show that $x = 0$ is a fixed point of the system, and show that the system can have up to three fixed points in total depending on the value of h. Identify the ranges of positive values of h on which the number of fixed points is constant.

[5 marks]

(b) Determine the stability of each of the fixed points identified in part (a).

[5 marks]

(c) Show that the fixed point at $x = 0$ undergoes a bifurcation at $h = a$ and determine the type of bifurcation.

[3 marks]

(d) Show that a bifurcation can occur at a fixed point $x > 0$ and determine the associated critical value of h and the type of bifurcation.

[3 marks]

4. (a) **Covariance functions.** You can use any results from the lecture and example sheet to address the following points:

(i) Consider n functions $\phi_i : \mathbb{R} \to \mathbb{R}$ ($i \in \{1, ..., n\}$). Assume $W_1, ..., W_n \overset{i.i.d.}{\sim} \mathcal{N}(1, 1)$ are identically, independently distributed with mean 1 and variance 1. For every $x \in \mathbb{R}$, define the random variable $F_x := f(x; W)$ where $f(x; W) = \sum_{i=1}^n W_i \phi_i(x)$ for $x \in \mathbb{R}$.

What are the mean and variance functions, $m(x) = \mathbb{E}[F_x] = \langle F_x \rangle$, and $v(x) = \mathrm{var}[F_x]$?

[4 marks]

(ii) Argue why $k : (x, \xi) \mapsto x\xi + \cos(x\xi - \pi)$ cannot be a valid covariance function on $\mathbb{R}^2$.

[2 marks]

(b) **Bayes nets.** Consider the Bayes net in Figure 1, representing the conditional independence structure of a joint probability distribution $P(A, B, C, D)$ of the discrete random variables $A, B, C,$ and D.

(i) Write down the factorisation of the joint distribution into conditional distributions warranted by the Bayes net, $P(A, B, C, D)$

(ii) Show the conditional independence of A and D given C,

$$P(A, D|C) = P(A|C)\, P(D|C)\,.$$

[5 marks]

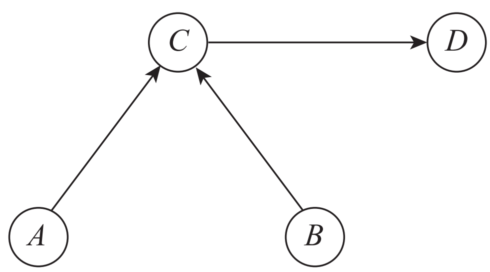

Figure 1

(c) **Gaussian processes– model comparison.** Consider a Bayesian regression problem for an uncertain function $f : \mathbb{R} \to \mathbb{R}$. Assume we have a noise-free sample $\mathcal{D} = \{(\xi, y)\}$ of f with input sample $\xi = 0$ and function value sample $y = f(\xi) = 0$.

(i) Consider a first GP prior model $\mathcal{M}_1 : f \sim \mathcal{GP}(\mu, k_{ff})$ with prior covariance function $k_{ff}(x, x') = \exp(-|x - x'|)$. Write down prior mean function $\mu : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{M}_1$ encodes the a priori expectation that f is a linear function intersecting with the origin and has a slope of 2.

[2 marks]

(ii) Consider a second GP prior $\mathcal{M}_2 : f \sim \mathcal{GP}(\tilde{\mu}, k_{ff})$ with the same covariance function but a different, constant prior mean function $\tilde{\mu}(x) = 1$. Calculate the log-likelihoods of the data to determine which of the two models is better supported by the data $\mathcal{D}$. For which does $\mathcal{D}$ provide more evidence, $\mathcal{M}_1$ or $\mathcal{M}_2$?

[3 marks]