

C24 2022 Solutions to Questions 2 & 3: Dynamical Systems (MRC)

2. (a) $\dot{y} = 0 \implies x = 0$ hence $(\dot{x}, \dot{y}) = (0, 0) \implies (x, y) = (0, 0)$. Linearize about $(0, 0)$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -y + \mu x \\ x \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} \mu & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{eigenvalues: } \lambda = \frac{1}{2}\mu \pm \frac{1}{2}(\mu^2 - 4)^{1/2}$$

Classification (using the Hartman-Grobman theorem):	$\mu < 0$	$\mu \leq -2$	$\text{Re}(\lambda) < 0$	$\text{Im}(\lambda) = 0$	stable node
		$-2 < \mu < 0$	$\text{Re}(\lambda) < 0$	$\text{Im}(\lambda) \neq 0$	stable spiral
	$\mu > 0$	$0 < \mu < 2$	$\text{Re}(\lambda) > 0$	$\text{Im}(\lambda) \neq 0$	unstable spiral
		$\mu \geq 2$	$\text{Re}(\lambda) > 0$	$\text{Im}(\lambda) = 0$	unstable node

(b) Assume $\mu < 0$.

The derivative of $V(x, y) = x^2 + y^2$ with respect to t is $\dot{V} = 2\mu x^2(1 - x^4/5)$.

Therefore $\mu < 0 \implies \dot{V} < -2|\mu|x^2$ if $|x| < 5^{1/4}$.

It follows that $\{(x, y) : x^2 + y^2 < 5^{1/2}\}$ is an invariant set.

LaSalle's Invariance Principle therefore implies that \dot{V} converges to 0 on every trajectory of the system starting from an initial condition in this set.

But $\dot{V} = 0$ implies $x = 0$, and $\dot{x} = 0$ therefore implies $y = 0$, so the largest invariant set on which $\dot{V} = 0$ consists of the origin.

Therefore any system trajectory with initial condition (x_0, y_0) satisfying $x_0^2 + y_0^2 < 5^{1/2}$ converges asymptotically to $(0, 0)$.

(c) Assume $\mu > 0$, then $(x, y) = (0, 0)$ is an unstable equilibrium point of the system (from the linearization analysis in part (a)).

Denote the set of points enclosed by C as S_C . Then S_C is necessarily invariant.

Also S_C contains no stable equilibrium points (since $(0, 0)$ is the only equilibrium point), and from the information in the question, S_C contains no limit cycles apart from C .

Therefore the Poincaré-Bendixson Theorem implies that all trajectories in S_C with non-zero initial conditions must converge to C .

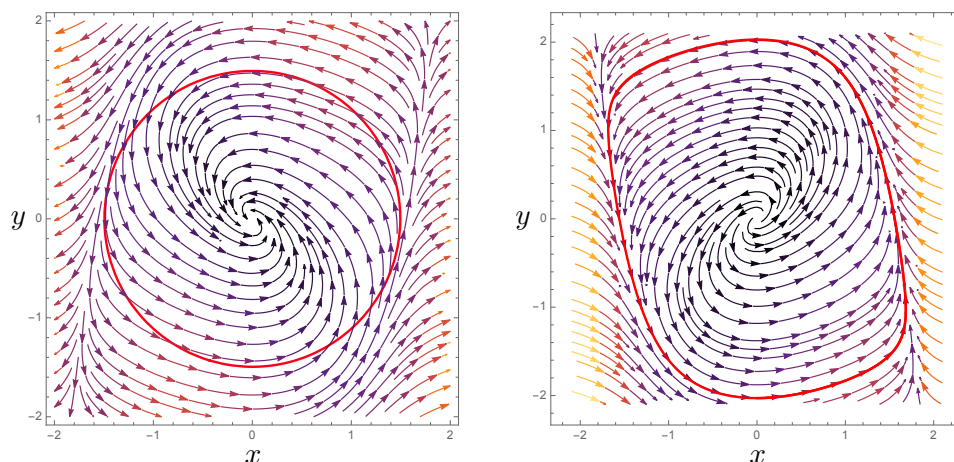


Figure 1: Left: phase portrait for $\mu = -1$ with $\{(x, y) : x^2 + y^2 = \sqrt{5}\}$ shown in red. Right: phase portrait for $\mu = 1$ with the limit cycle C shown in red.

3. (a) The fixed points are the values of x such that $f(x) = x(1 - x) - hx/(x + a) = 0$.

Since $x \geq 0$ and $a > 0$ we can multiply this equation by $x + a$ to get a cubic polynomial:

$$x(1 - x)(a + x) - hx = 0 \implies x = 0 \text{ or } (1 - x)(a + x) - h = 0$$

$$x = 0 \text{ or } x = \frac{1}{2}(1 - a) \pm \left[\frac{1}{4}(1 + a)^2 - h\right]^{1/2}$$

Considering the behaviour of the roots as h increases from 0, the fixed points are

$$x = 0 \text{ or } x = \frac{1}{2}(1 - a) \pm \left[\frac{1}{4}(1 + a)^2 - h\right]^{1/2} \text{ if } h < \frac{1}{4}(1 + a)^2 \implies 3 \text{ fixed points}$$

$$x = 0 \text{ or } x = \frac{1}{2}(1 - a) \text{ if } h = \frac{1}{4}(1 + a)^2 \implies 2 \text{ fixed points}$$

$$x = 0 \text{ if } h > \frac{1}{4}(1 + a)^2 \implies 1 \text{ fixed point}$$

(It could also be noted that x is proportional to density, and we must therefore have $x \geq 0$, but only 2 out of the 3 fixed points satisfy $x \geq 0$ when $h < a$.)

- (b) To find the stability of each fixed point consider the sign of $f'(x) = 1 - 2x - ah/(x + a)^2$:

$$x = 0 \implies f'(0) = 1 - h/a \begin{cases} > 0 \implies \text{unstable if } h < a \\ < 0 \implies \text{stable if } h > a \end{cases}$$

$$x = \frac{1}{2}(1 - a) - \left[\frac{1}{4}(1 + a)^2 - h\right]^{1/2} \implies f'(x) \begin{cases} > 0 \implies \text{unstable if } 0 < h < a \\ < 0 \implies \text{stable if } a < h < \frac{1}{4}(1 + a)^2 \end{cases}$$

$$x = \frac{1}{2}(1 - a) + \left[\frac{1}{4}(1 + a)^2 - h\right]^{1/2} \implies f'(x) < 0 \implies \text{stable if } 0 < h < \frac{1}{4}(1 + a)^2$$

Here the sign of $f'(x)$ can be determined using the continuity of $f(x)$ and $f'(0) > 0$ for $h < a$ and $f'(0) < 0$ for $h > a$, or alternatively by using a sketch of $f(x)$ (Fig. 2).

For $h = \frac{1}{4}(1 + a)^2$ the equilibrium point at $x = \frac{1}{2}(1 - a)$ satisfies $f'(x) = 0$ and $f''(x) > 0$, and is therefore a saddle point.

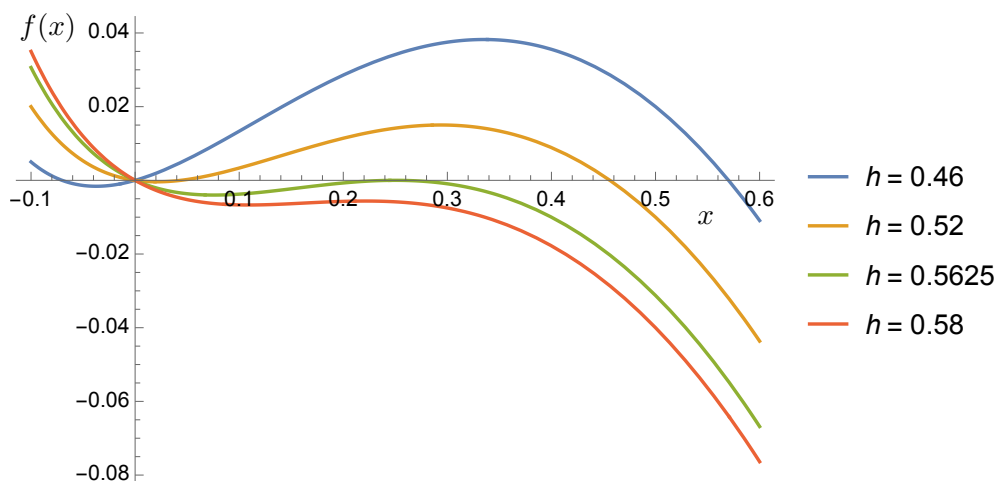


Figure 2: $f(x)$ for $a = 0.5$ and $\frac{1}{4}(1 + a)^2 = \frac{9}{16} = 0.5625$

The location and stability of each equilibrium point could be shown on a bifurcation diagram (although this is not asked for in the question) as shown in Fig. 3.

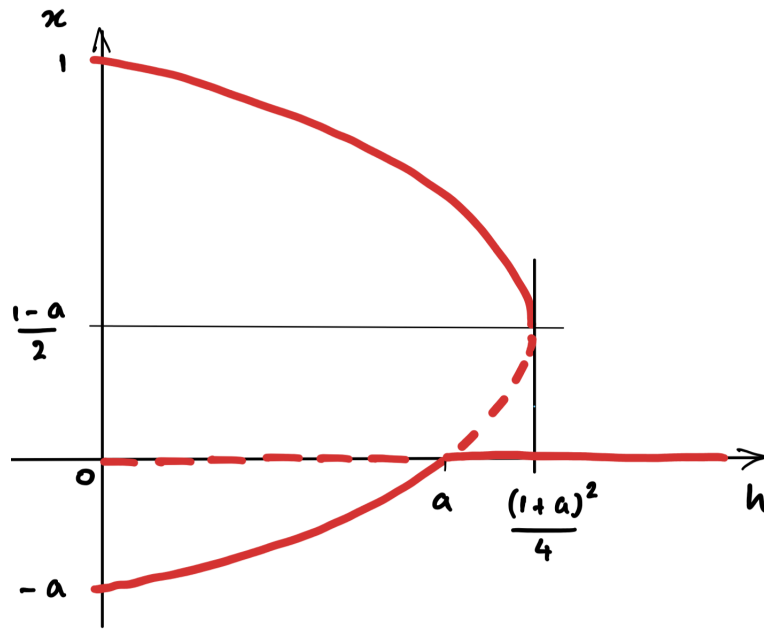


Figure 3: Bifurcation diagram showing stable equilibria (red solid lines) and unstable equilibria (red dashed lines).

(c) Let $f(x, h) = x(1 - x) - hx/(x + a) = 0$. Tangency conditions at $x_0 = 0$, $h_0 = a$:

$$\begin{aligned} \frac{\partial f}{\partial x} \Big|_{x_0, h_0} &= 1 - 2x_0 - ah_0/(x_0 + a)^2 = 0, & \frac{\partial f}{\partial h} \Big|_{x_0, h_0} &= -x_0/(x_0 + a) = 0 \\ \frac{\partial^2 f}{\partial x \partial h} \Big|_{x_0, h_0} &= -a/(x_0 + a)^2 = -1/a \neq 0 \\ \frac{\partial^2 f}{\partial x^2} \Big|_{x_0, h_0} &= -2 + 2ah_0/(x_0 + a)^3 = 2(1 - a)/a \neq 0 \end{aligned}$$

This a transcritical bifurcation.

(d) Tangency conditions for the bifurcation at $x_0 = \frac{1}{2}(1 - a)$ and $h_0 = \frac{1}{4}(1 + a)^2$:

$$\begin{aligned} \frac{\partial f}{\partial x} \Big|_{x_0, h_0} &= 1 - 2x_0 - ah_0/(x_0 + a)^2 = 0 \\ \frac{\partial f}{\partial h} \Big|_{x_0, h_0} &= -x_0/(x_0 + a) = -(1 - a)/(1 + a) \neq 0 \\ \frac{\partial^2 f}{\partial x^2} \Big|_{x_0, h_0} &= -2 + 2ah_0/(x_0 + a)^3 = -2(1 - a)/(1 + a) \neq 0 \end{aligned}$$

This a saddle node bifurcation.