A10613W1

TRINITY TERM 2023

SECOND PUBLIC EXAMINATION

Honour School of Engineering Science

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (Paper C24)

Wednesday 7 June 2023 9:30 a.m. – 11:00 a.m.

Answers to not more than **THREE** *questions should be submitted.*

Answer each question in a separate booklet.

DO NOT TURN THIS PAGE UNTIL TOLD YOU MAY DO SO

Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*

1. Consider the boundary-value problem

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\epsilon \frac{\mathrm{d}y}{\mathrm{d}x} - xy = -x \quad \text{with} \quad y(0) = 0\ ,\ y(1) = 2\,.$$

(a) Use a standard perturbation expansion $y = y_0 + \epsilon y_1 + \ldots$ to show that there must be two boundary layers. Derive the leading-order outer solution.

[3 marks]

(b) Introduce an appropriately stretched coordinate $\bar{x}$ for the boundary layer *on the left* and derive the associated leading-order inner solution $\bar{Y}(\bar{x})$, applying the relevant boundary condition.

[4 marks]

(c) Introduce an appropriate stretched coordinate $\hat{x}$ for the boundary layer *on the right* and derive the associated leading-order inner solution $\hat{Y}(\hat{x})$, applying the relevant boundary condition.

[5 marks]

(d) Identify appropriate matching conditions and use them to match the two inner solutions to the outer solution. Thus, construct a composite solution. Sketch the composite solution, indicating key features.

[4 marks]

2. An ecosystem sustains populations of foxes and rabbits with sizes $f(t)$ and $r(t)$, respectively. In the absence of rabbits, the ecosystem has a carrying capacity $f_{\max}$ for foxes; in the absence of foxes, the ecosystem has a carrying capacity $r_{\max}$ for rabbits. Since foxes eat rabbits, the fox population grows when more rabbits are available to eat; similarly, the population of rabbits shrinks if there are more foxes. Letting $F = f/f_{\max}$ and $R = r/r_{\max}$, this coupled predator-prey interaction is captured by the dynamical system

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{F(1-F)}{4} + \frac{FR}{10}$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = 2R(1-R) - aFR\,,$$

where t is the time in years, and a is a positive parameter that quantifies the annual rate of rabbit predation by the fox population.

(a) This autonomous system has four equilibrium points. Determine the vectors $(F, R)^*$ at each of these equilibria.

[3 marks]

(b) Assuming that $a = 1$, determine the Jacobian matrix of the autonomous system. Show that all four equilibrium points identified in part (a) are hyperbolic in this instance. Hence characterise the four equilibria as stable nodes, unstable nodes, or saddle points.

[6 marks]

(c) Sketch the phase portrait of the autonomous system when $a = 1$. Use Dulac's criterion with weighting function $B(F, R) = 1/(FR)$ to assess whether closed orbits can exist in the quadrant of the phase plane where both the fox and rabbit populations are nonzero. What does this conclusion suggest about the stable equilibrium point (or points) you identified in part (b)?

[5 marks]

(d) Determine the maximum value of a that allows a rabbit population to survive over the long term in the presence of foxes.

[2 marks]

3. The motion of a mass attached to a spring–damper system follows the dynamical equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2\sinh^2\left(\tfrac{1}{2}x\right) - \gamma x^2 v\,,$$

where x is the position of the mass, v is its velocity, and $\gamma \geq 0$ is a drag factor.

(a) Without damping, $\gamma = 0$. Show that when the dynamical equations above are undamped, they are a Hamiltonian system. Do this by deriving a Hamiltonian function $H(x, v)$ for the system with $\gamma = 0$. Write the Hamiltonian function that satisfies $H(0, 0) = 0$.

[4 marks]

(b) Again assuming the system is undamped ($\gamma = 0$), calculate the time derivative of the Hamiltonian function you found in part (a). In light of this result, what does the Poincaré–Bendixson theorem imply about trajectories that do not pass through the origin of the phase plane?

[4 marks]

(c) Consider the damped situation, with $\gamma > 0$ in the dynamical equations at the start of the question. Propose a Lyapunov function and use it to show that the origin is a stable equilibrium point.

[6 marks]

(d) Further explain whether or not the origin is asymptotically stable when $\gamma > 0$.

[2 marks]

4. (a) You can use any results from the lecture and example sheet to address the following points:

(i) Consider the Bayesian linear regression model prior over function values $F_x := f(x; W) = W_1 x + W_2 x^2$ for real-valued inputs x and uncertain parameter $W \sim \mathcal{N}(0, 2I)$ where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix. Under this prior model, let $m(x) = \mathbb{E}[F_x]$ denote the mean of F_x and $v(x) = k(x, x) = var[F_x]$ the variance.

Determine $m(2)$ and $v(2)$.

[2 marks]

(ii) Decide whether $k : (x, \xi) \mapsto x\xi + \cos(|x - \xi| - \pi)$ can be a valid covariance function on $\mathbb{R}^2$ and explain your choice.

[1 mark]

(iii) Assume you are given a function $k : \mathbb{R}^2 \to \mathbb{R}$ with $k(2, 2) = k(1, 1) = 1$, $k(1, 2) = 2$. Appeal to Cauchy-Schwarz inequality to show why k cannot be a valid covariance function.

[2 marks]

(b) Consider the Bayes net in Figure 1, representing the conditional independence structure of a joint probability distribution $P(A, B, C, D)$ of the discrete random variables A, B, C, and D.

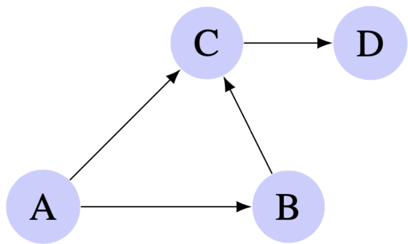

Figure 1

Write down the factorisation of the joint distribution into conditional distributions warranted by the Bayes net: $P(A, B, C, D) = ...?$

[2 marks]

(c) Consider a one-dimensional Bayesian regression problem where the uncertain function we wish to infer is $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and where we know that it is a solution to the linear differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = -f(t), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

A priori, we assume the initial value $w := f(0)$ is uncertain and modelled as Gaussian distributed random variable $W \sim \mathcal{N}(0, 1)$.

(i) Argue why this uncertainty about the initial value corresponds to a prior Gaussian process $F \sim \mathcal{GP}(\mu, k)$ about f with $F_t = \mathrm{e}^{-t}\, W$ and determine its prior mean function $\mu : t \mapsto \mathbb{E}[F_t]$ and covariance function $k : (t, t') \mapsto \mathrm{cov}(F_t, F_{t'})$.

[3 marks]

(ii) Suppose that, for time $\tau = 1$, we were able to observe a noisy sample $\mathcal{D} = \{(\tau, y)\}$ where noisy function value $y = f(\tau) + \nu = 1$ and the noise perturbation ν was drawn from a standard normal distribution: $\nu \sim \mathcal{N}(0, 1)$. Determine a maximum likelihood estimate w_{ml} of the initial value parameter w.

[3 marks]

(iii) Furthermore, determine a maximum a posteriori estimate w_{map} of the initial value parameter w.

[3 marks]

Note: In addition to providing numerical values, make sure to show how you derived them.