C24 Perturbation Methods Exam 2022-23

$$\varepsilon^2 \frac{d^2y}{dx^2} + 2\varepsilon \frac{dy}{dx} - xy = -x \qquad y(0) = 0, \quad y(1) = 2$$

(a) let $y = y_0 + \varepsilon y_1 + \dots$

$$\varepsilon^2 (y_0'' + \varepsilon y_1'' + \dots) + 2\varepsilon (y_0' + \varepsilon y_1' + \dots) - x(y_0 + \varepsilon y_1 + \dots) = -x$$ [1]

ε^0: $\quad -x y_0 = -x \rightarrow y_0(x) = 1$ [1]

Cannot satisfy either of the BCs! Thus, there must be BLs @ both boundaries and this is our outer solution (valid in the middle) [1]

(b) On the left: $\bar{y}(\bar{x}) = y(x)$, $\bar{x} = x/\varepsilon$ (zoom around $x=0$) [1]

$\frac{d}{dx} = \varepsilon^{-1} \frac{d}{d\bar{x}}$, $x = \varepsilon \bar{x}$: $\quad \bar{y}'' + 2\bar{y}' - \varepsilon \bar{x} \bar{y} = -\varepsilon \bar{x}$ [1]

let $\bar{y} = \bar{y}_0 + \varepsilon \bar{y}_1 + \dots$

$$(\bar{y}_0'' + \varepsilon \bar{y}_1'' + \dots) + 2(\bar{y}_0' + \varepsilon \bar{y}_1' + \dots) - \varepsilon \bar{x}(\bar{y}_0 + \varepsilon \bar{y}_1 + \dots) = -\varepsilon \bar{x}$$

ε^0: $\quad \bar{y}_0'' + 2\bar{y}_0' = 0$ [1]

$\bar{y}_0 = \bar{A} + \bar{B} e^{-2\bar{x}}$

BC: $y(0) = 0 \rightarrow \bar{y}(0) = 0$: $\quad \bar{y}_0(0) = \bar{A} + \bar{B} = 0 \rightarrow \bar{B} = -\bar{A} \rightarrow \bar{y}_0(\bar{x}) = \bar{A}(1 - e^{-2\bar{x}})$ [1]

(c) On the right: $\hat{y}(\hat{x}) = y(x)$, $\hat{x} = \frac{1-x}{\varepsilon}$ (zoom around $x=1$) [1]

$\frac{d}{dx} = -\varepsilon^{-1} \frac{d}{d\hat{x}}$, $x = 1 - \varepsilon \hat{x}$: $\quad \hat{y}'' - 2\hat{y}' - (1 - \varepsilon \hat{x})\hat{y} = -(1 - \varepsilon \hat{x})$ [1]

let $\hat{y} = \hat{y}_0 + \varepsilon \hat{y}_1 + \dots$

$$(\hat{y}_0'' + \varepsilon \hat{y}_1'' + \dots) - 2(\hat{y}_0' + \varepsilon \hat{y}_1' + \dots) - (1 - \varepsilon \hat{x})(\hat{y}_0 + \varepsilon \hat{y}_1 + \dots) = -(1 - \varepsilon \hat{x})$$

ε^0: $\quad \hat{y}_0'' - 2\hat{y}_0' - \hat{y}_0 = -1$ [1]

roots: $\lambda^2 - 2\lambda - 1 = 0$

$\lambda = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2} \rightarrow \hat{y}_0$

$$\hat{y}_0 = \underbrace{\hat{A} e^{(1+\sqrt{2})\hat{x}} + \hat{B} e^{(1-\sqrt{2})}}_{\text{complementary fn.}} + \underbrace{1}_{\text{particular integral}}$$

BC: $y(1)=2 \to \hat{y}(0)=2$

$\hat{y}_0(0) = \hat{A} + \hat{B} + 1 = 2 \to \hat{B} = 1-\hat{A}$

$$\boxed{\hat{y}_0(\hat{x}) = \hat{A}e^{(1+\sqrt{2})\hat{x}} + (1-\hat{A})e^{(1-\sqrt{2})\hat{x}} + 1}$$ [1] [1]

(d) Match @ left:

$$\lim_{x\to 0} y_0(x) = \lim_{\bar{x}\to\infty} \bar{y}_0(\bar{x})$$

$$1 = \lim_{\bar{x}\to\infty} \bar{A}(1-\cancel{e^{-2\bar{x}}}^{\,0}) = \bar{A}$$

$$\boxed{\bar{y}_0(\bar{x}) = 1-e^{-2\bar{x}}}$$ [1]

$(\bar{y}_{overlap} = 1)$

Match @ right:

$$\lim_{x\to 1} y_0(x) = \lim_{\hat{x}\to\infty} \hat{y}_0(\hat{x})$$

$\to 0$ as $\hat{x}\to\infty$

$$1 = \lim_{\hat{x}\to\infty} \hat{A}\cancel{e^{(1+\sqrt{2})\hat{x}}} + (1-\hat{A})\cancel{e^{(1-\sqrt{2})\hat{x}}} + 1 = 1 \checkmark$$

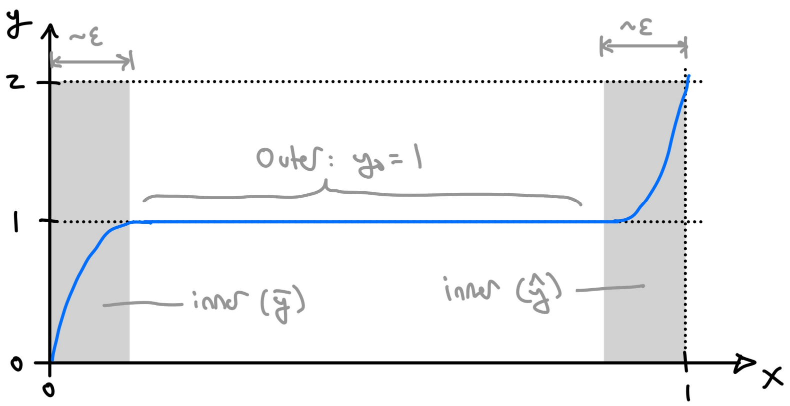
$\boxed{\text{Need } \hat{A}=0 \text{ to avoid divergence.}}$

$$\boxed{\hat{y}_0(\hat{x}) = 1+e^{(1-\sqrt{2})\hat{x}}}$$ [1]

$(\hat{y}_{overlap} = 1)$

Composite:
$$y(x) = y_0(x) + \bar{y}_0(\bar{x}) - \bar{y}_{overlap} + \hat{y}_0(\hat{x}) - \hat{y}_{overlap}$$
$$= 1 + \cancel{1}-e^{-2\bar{x}} - \cancel{1} + \cancel{1}+e^{(1-\sqrt{2})\hat{x}} - \cancel{1}$$

$$\boxed{y(x) = 1-e^{-2x/\varepsilon} + e^{-(\sqrt{2}-1)(1-x)/\varepsilon}}$$ [1]



[1]

C24 Dynamical systems

Q1 (a) To find equilibria set $\frac{dF}{dt} = \frac{dR}{dt} = 0$

$$0 = F\left[\tfrac{1}{4}(1-F) + \tfrac{1}{10}R\right] \quad (1)$$

$$0 = R\left[2(1-R) - aF\right] \quad (2)$$

(1) satisfied if $F=0$ or $\frac{1}{4}(1-F) + \frac{1}{10}R = 0$

(2) satisfied if $R=0$ or $2(1-R) - aF = 0$

If $F=0$ then equilibria if $R=0$ or 1

If $R=0$ then equilibria if $F=0$ or 1

If $F \neq 0$ and $R \neq 0$ then equilibrium if

$$\begin{aligned} \tfrac{1}{4}F - \tfrac{1}{10}R &= \tfrac{1}{4} \\ aF + 2R &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 10 & -4 \\ a & 2 \end{bmatrix}\begin{bmatrix} F \\ R \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} F \\ R \end{bmatrix} = \frac{1}{20+4a}\begin{bmatrix} 2 & 4 \\ -a & 10 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{5+a} \\ \frac{10-5a}{10+2a} \end{bmatrix}$$

Equilibria are $\boxed{\begin{bmatrix} F \\ R \end{bmatrix}^{*} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{5+a} \\ \frac{10-5a}{10+2a} \end{bmatrix} \right\}}$

(b)

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial \dot{F}}{\partial F} & \frac{\partial \dot{F}}{\partial R} \\ \frac{\partial \dot{R}}{\partial F} & \frac{\partial \dot{R}}{\partial R} \end{bmatrix} \qquad \frac{\partial \dot{F}}{\partial F} = \tfrac{1}{4}(1-F) + \tfrac{1}{10}R - \tfrac{1}{4}F \qquad \frac{\partial \dot{F}}{\partial R} = \tfrac{1}{10}F$$

$$\frac{\partial \dot{R}}{\partial F} = -aR \qquad \frac{\partial \dot{R}}{\partial R} = 2(1-R) - aF - 2R$$

with $a=1$

$$J(0,0) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 2 \end{bmatrix} \quad J(0,1) = \begin{bmatrix} \frac{7}{20} & 0 \\ -1 & -2 \end{bmatrix} \quad J(1,0) = \begin{bmatrix} -\frac{1}{4} & \frac{1}{10} \\ 0 & 1 \end{bmatrix}$$

$$J\left(\tfrac{7}{6}, \tfrac{5}{12}\right) = \begin{bmatrix} \frac{1}{4} - \frac{7}{24} + \frac{5}{120} - \frac{7}{24} & \frac{7}{60} \\ -\frac{5}{12} & \frac{7}{6} - \frac{7}{6} - \frac{5}{6} \end{bmatrix} = \begin{bmatrix} -\frac{7}{24} & \frac{7}{60} \\ -\frac{5}{12} & -\frac{5}{6} \end{bmatrix}$$

Q1 (b) cont'd

Eigenvalues $J(0,0) \Rightarrow \{\frac{1}{4}, 2\}$ unstable node

Eigenvalues $J(0,1) \Rightarrow \{\frac{7}{20}, -2\}$ saddle point

Eigenvalues $J(1,0) \Rightarrow \{-\frac{1}{4}, 1\}$ saddle point

Eigenvalues $J(\frac{7}{6}, \frac{5}{12}) \Rightarrow (-\frac{7}{24} - \lambda)(-\frac{5}{6} - \lambda) + \frac{35}{720} = 0$

$5(7 + 24\lambda)(5 + 6\lambda) + 35 = 0$

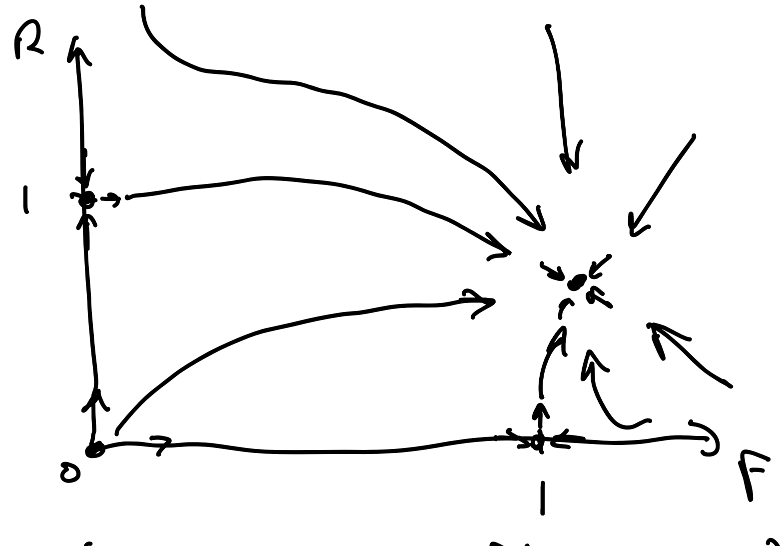
$(7+24\lambda)(5+6\lambda) + 7 = 35 + 42\lambda + 120\lambda + 144\lambda^2 + 7 = 0$

$144\lambda^2 + 162\lambda + 42 = 0$

$\{-\frac{1}{48}(27+\sqrt{57}), -\frac{1}{48}(27-\sqrt{57})\} = \{-0.7198\ldots, -0.4052\ldots\}$

stable node

(c)



Dulac $\Rightarrow \frac{\partial}{\partial F}\left[\frac{F(1-F)}{4FR} + \frac{1}{10}\right] + \frac{\partial}{\partial R}\left[\frac{2R(1-R)}{FR} - a\right]$

$= -\frac{1}{4R} - \frac{2}{F}$ Negative in upper right quadrant so no closed orbits there

(d) Argument in (c) holds for any a value. Rabbits die out if $a \geq 2$, which moves stable node to lower right quadrant.

C24 Dynamical Systems

(Q2) (a) For equations to be a Hamiltonian system we need a function $H(x,v)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial v} \qquad \Rightarrow \qquad \frac{\partial H}{\partial v} = v \quad \text{so} \quad H(x,v) = \tfrac{1}{2}v^2 + f(x)$$

$$\frac{dv}{dt} = -\frac{\partial H}{\partial x} \qquad\qquad \frac{\partial H}{\partial x} = 2\sinh^2\left(\tfrac{x}{2}\right) = \cosh(x) - 1$$

$$\text{so } H(x,v) = \sinh x - x + g(v)$$

Thus $H(x,v) = \frac{1}{2}v^2 + \sinh x - x + k$

and $H(0,0) = 0$ so $\boxed{H(x,v) = \frac{1}{2}v^2 + \sinh x - x}$

(b) Taking the function from part (a)

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt} = (\cosh x - 1)v + v\left[-2\sinh^2\left(\tfrac{1}{2}x\right)\right]$$

$$= 2v\sinh^2\left(\tfrac{x}{2}\right) - 2v\sinh^2\left(\tfrac{x}{2}\right) = 0$$

We have a function $H(x,v)$ such that $\frac{dH}{dt} = 0$ for all (x,v). There is only a single equilibrium point in $\mathbb{R}^2$, at the origin. Because $\frac{dH}{dt} = 0$ everywhere, all of $\mathbb{R}^2$ is positively invariant. Poincaré-Bendixson says trajectories that do not pass through the origin must be closed orbits.

Q2 (b) cont'd

Also consider that the Jacobian at the origin is

$J(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; it is not a hyperbolic point.

This further supports that the closed orbits are stable limit cycles.

(c) Use the Hamiltonian from (a) as a Lyapunov function. Then

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial x}\right)\dot{x} + \left(\frac{\partial H}{\partial v}\right)\dot{v} = 2v\sinh^2\left(\frac{x}{2}\right) + v\left[-2\sinh^2\left(\frac{x}{2}\right) - \gamma x^2 v\right]$$

$$\frac{dH}{dt} = -\gamma x^2 v^2$$

We have that $\frac{dH}{dt} \leq 0$ for all $\begin{bmatrix} x \\ v \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so the origin is stable.

(d) Global asymptotic stability requires that $\lim_{\|\vec{x}\| \to \infty} \frac{dH}{dt} = \infty$. This is not true for our function

$\frac{dH}{dt} = -\gamma x^2 v^2$, for $\|\vec{x}\| = \sqrt{x^2 + v^2}$ when $\vec{x}$ is away from the origin and orientated such that $v = 0$ or $x = 0$.

(a) Covariance functions. [5 mark(s)]

You can use any results from the lecture and example sheet to address the following points:

1. Consider the Bayesian linear regression model prior over function values $F_x := f(x; W) = W_1 x + W_2 x^2$ for real-valued inputs x and uncertain parameter $W \sim \mathcal{N}(0, 2I)$ where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix. Under this prior model, let $m(x) = \mathbb{E}[F_x]$ denote the mean of F_x and $v(x) = k(x, x) = var[F_x]$ the variance. Determine $m(2)$ and $v(2)$. **[2 mark(s)]**

2. Decide whether $k : (x, \xi) \mapsto x\xi + \cos(|x - \xi| - \pi)$ can be a valid covariance function on $\mathbb{R}^2$ and explain your choice." **[1 mark(s)]**

3. Assume you are given a function $k : \mathbb{R}^2 \to \mathbb{R}$ with $k(2, 2) = k(1, 1) = 1$, $k(1, 2) = 2$. Appeal to Cauchy-Schwarz inequality to show why k cannot be a valid covariance function. **[2 mark(s)]**

Answers.

1. In the lecture, for uncorrelated W_i and Bayesian models of the form $F_x = \sum_{i=1}^n W_i \phi_i(x)$, we have derived the formula $v(x) = \sum_{i=1}^n var[W_i]\phi_i^2(x)$. Plugging this in with $n = 1, \phi_1(x) = x, \phi_2(x) = x^2$ we get

$$v(x) = 2\,x^2 + 2\,x^4.$$

Hence,

$$v(2) = 8 + 32 = 40.$$

[1 mark(s)] Furthermore, using the linearity of the expectation operator, we get $m(x) = 0\,\forall x$, and thus,

$$m(2) = 0.$$

[1 mark(s)]

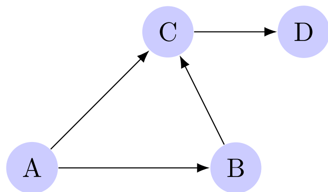
2. $k(0, 0) = -1$. Hence, k cannot be a variance (i.e. k is not pos. def.) and k is not a valid covariance function. **[1 mark(s)]**

3. Indeed, k cannot be a covariance function since due to Cauchy-Schwartz we must have $k(x, x')^2 \leq k(x, x)k(x, x')$ for all inputs x, x'. However, for $x = 1, x' = 2$, we have

$$k(x, x')^2 = 4 > 1 = k(x, x)k(x', x').$$

[2 mark(s)]

(b) Bayes nets. [2 mark(s)]



Consider the Bayes net in the picture, representing the conditional independence structure of a joint probability distribution $P(A, B, C, D)$ of the discrete random variables A, B, C, D.

1. Write down the factorisation of the joint distribution into conditional distributions warranted by the Bayes net: $P(A, B, C, D) = ...?$ **[2 mark(s)]**

Answers.

1. $P(A, B, C, D) = P(A)P(B|A)P(C|A, B)P(D|C)$ **[2 mark(s)]**

(c) Bayesian regression. [9 mark(s)]

Consider a one-dimensional Bayesian regression problem where the uncertain function we wish to infer is $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and where we know that it is a solution to the linear differential equation

$$\frac{df}{dt}(t) = -f(t), \forall t \in \mathbb{R}_{\geq 0}.$$

A priori, we assume the initial value $w := f(0)$ is uncertain and modelled as Gaussian distributed random variable $W \sim \mathcal{N}(0, 1)$.

1. Argue why this uncertainty about the initial value corresponds to a prior Gaussian process $F \sim \mathcal{GP}(\mu, k)$ about f with $F_t = e^{-t}W$ and determine its prior mean function $\mu : t \mapsto \mathbb{E}[F_t]$ and covariance function $k : (t, t') \mapsto \mathrm{cov}(F_t, F_{t'})$. **[3 mark(s)]**

2. Suppose that, for time $\tau = 1$, we were able to observe a noisy sample $\mathcal{D} = \{(\tau, y)\}$ where noisy function value $y = f(\tau) + \nu = 1$ and the noise perturbation ν was drawn from a a standard normal distribution: $\nu \sim \mathcal{N}(0, 1)$. Determine a maximum likelihood estimate w_{ml} of the initial value parameter w. **[3 mark(s)]**

3. Furthermore, determine a maximum a posteriori estimate w_{map} of the initial value parameter w. **[3 mark(s)]**

Make sure to not just provide the numerical values but also to show how you derived them.

Answers.

1. Every solution of the ODE is of the form $f(t) = we^{-t}$ where $w = f(0)$. Since $W \sim \mathcal{N}(0,1)$, $F_t = We^{-t}$ is a Gaussian random variable with mean $\mu(t) = 0$. In addition $k(t,t') = \operatorname{cov}(F_t, F_{t'}) = \operatorname{cov}(W,W)e^{-t}e^{-t'} = e^{-(t+t')}$. **[3 mark(s)]**

2. Likelihood:

$$\begin{aligned}\underbrace{p(y|\tau,w)}_{\text{Gaussian density}} &\propto \exp(-\frac{1}{2}(y - f(\tau))^2) \\ &= \exp(-\frac{1}{2}(1 - we^{-1})^2)\end{aligned}$$

Hence:

$$\begin{aligned} w_{ml} &\in \arg\max_w \log\left(p(y|\tau,w)\right) \\ &= \arg\max_w -\frac{1}{2}(1 - we^{-1})^2 \\ &= \arg\min_w (1 - we^{-1})^2 \qquad = \{e\}. \end{aligned}$$

[3 mark(s)]

3. MAP:

$$\begin{aligned} w_{map} &\in \arg\max_w \log\left(p(y|\tau,w)p(w)\right) \\ &= \arg\max_w -\frac{1}{2}(1 - we^{-1})^2 - \frac{1}{2}w^2 \\ &= \arg\min_w \underbrace{(1 - we^{-1})^2 + w^2}_{=:g(w)} \end{aligned}$$

Dealing with a (convex) quadratic function g, all we need to do is to solve $g'(w_{map}) = 0$:

$$\begin{aligned} g'(w_{map}) &= -2e^{-1}(1 - w_{map}e^{-1}) + 2w_{map} \overset{!}{=} 0 \\ \Leftrightarrow & e^{-1} - w_{map}e^{-2} = w_{map} \\ \Leftrightarrow & e^{-1} = w_{map}(1 + e^{-2}). \end{aligned}$$

Hence

$$w_{map} = \frac{e^{-1}}{1 + e^{-2}} \approx 0.324.$$

[3 mark(s)]