A10613W1

TRINITY TERM 2024

SECOND PUBLIC EXAMINATION

Honour School of Engineering Science

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (Paper C24)

Wednesday 12th June 2024 9.30 a.m. – 11 a.m.

Answers to not more than **THREE** *questions should be submitted.*

Answer each question in a separate booklet.

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Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*

1. (a) Show that the origin $(x_1, x_2) = (0, 0)$ is an equilibrium of the system defined by

$$\dot{x}_1 = -x_1 - x_2/\ln(r)$$
$$\dot{x}_2 = -x_2 + x_1/\ln(r)$$
$$r = \sqrt{x_1^2 + x_2^2},$$

and show that there are no other equilibrium points for $r < 1$.

[4 marks]

(b) Linearise the system in part (a) about the origin. Sketch the phase portrait of the linearized system and classify the type of equilibrium.

[6 marks]

(c) Show that the time-derivatives of the polar coordinates (r, θ) where $r^2 = x_1^2 + x_2^2$ and $\tan(\theta) = x_2/x_1$, are given by

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$r^2\dot{\theta} = x_1\dot{x}_2 - x_2\dot{x}_1.$$

[2 marks]

(d) Transform the system in (a) into polar coordinates and sketch the nonlinear system phase portrait for $r < 1$. Is this phase portrait consistent with your answer to part (b)?

[4 marks]

2. Let $\boldsymbol{x} = (x_1, x_2)$ be the state of a second order system satisfying

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g(x_1)$$

where g is a continuously differentiable function satisfying $zg(z) > 0$ for all $z \neq 0$ and $\int_0^y g(z)\,dz \to \infty$ as $y \to \infty$. Let V be the function defined by

$$V(\boldsymbol{x}) = \tfrac{1}{2}x_2^2 + \int_0^{x_1} g(z)\,\mathrm{d}z.$$

(a) Show that $V(\boldsymbol{x})$ remains constant along the trajectories of the system.

[2 marks]

(b) For any constant $c > 0$, show that $\Gamma = \{\boldsymbol{x} : V(\boldsymbol{x}) = c\}$ is a closed curve. Explain why every solution of the system must be periodic.

[6 marks]

(c) Now consider the modified dynamics

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g(x_1) + \epsilon x_2$$

where $\epsilon \neq 0$ is a constant. Can this system have periodic solutions? Explain your reasoning.

[8 marks]

3. Consider the boundary-value problem (BVP):

$$\epsilon \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0 \quad \text{with} \quad y(0) = 1\ ,\ y(1) = 0.$$

(a) Use a standard perturbation expansion $y = y_0 + \epsilon y_1 + \ldots$ to show that there must be a boundary layer. Identify and explain the location of the boundary layer. Derive the leading-order outer solution.

[2 marks]

(b) Introduce an appropriate stretched coordinate $\bar{x}$ and derive the associated leading-order inner solution $\bar{Y}(\bar{x})$, applying the relevant boundary condition.

[2 marks]

(c) Identify an appropriate matching condition and use it to match your outer solution from part (a) to your inner solution from part (b). Thus, construct a composite solution. Sketch the composite solution, indicating key features.

[3 marks]

(d) This problem can also be solved using the method of multiple scales. Rewrite the original BVP in terms of *two* independent variables, $y = y(x_1, x_2)$ with $x_1 = x$ and $x_2 = \bar{x}$, where $\bar{x}$ is the same as in part (b) above.

[2 marks]

(e) Introduce a standard perturbation expansion $y = y_0 + \epsilon y_1 + \ldots$ into your transformed BVP from part (d) and then identify and solve the leading-order problem.

[3 marks]

(f) Identify the next-order problem in your expanded equation from part (e) and use it to construct a complete leading-order solution. Compare this solution to your composite solution from part (c).

[4 marks]

4. (a) Suppose the price of one unit of a crypto-asset, *BayesCoin*, follows a trajectory given by the function $f : \mathbb{R} \to \mathbb{R}$ where $f(t)$ denotes the price at time t. Represent the Bayesian *a priori* belief about the trajectory by a Gaussian process. In particular, assume *a priori* that $f \sim \mathcal{GP}(\mu, k)$ where prior mean function $\mu(t) = 0$, and covariance function, k, is given by $k(t, t') = \exp(-|t - t'|)$ for $t, t' \in \mathbb{R}$.

Suppose that, at time $t_0 = 0$, it is observed that the current price is $f(0) = 2$. Using Bayesian methods, given this new information, what would you expect the price to be at future time $t = 1$? Give an approximate numerical value.

[3 marks]

(b) Consider a noisy sample $D = \{(x_1, y_1), (x_2, y_2)\}$ of a real-valued function $f : x \mapsto 2x$. Assume $y_i = f(x_i) + v_i$, where the (unknown) noise perturbations, v_i, are drawn independently from the same Gaussian distribution $\mathcal{N}(0, a^{-1})$. Suppose the following values are observed:

$x_1 = 0$; $y_1 = 1$;
$x_2 = 2$; $y_2 = 0$.

Assume that the precision (inverse variance) parameter a is uncertain and needs to be estimated. Also assume that the *a priori* belief about this parameter is modelled by an exponentially distributed random variable A with probability density function $p(A = a) = 3\exp(-3a)$. Derive and give approximate numerical values of the following:

(i) A maximum likelihood estimate $a_{\text{ml}} \in \arg\max_{a \in (0,\infty)} p(y_1, y_2 \,|\, x_1, x_2, a)$ of the precision a.

(ii) A maximum *a posteriori* estimate $a_{\text{map}} \in \arg\max_{a \in (0,\infty)} p(A = a \,|\, D)$ of the precision.

[6 marks]

(c) Explain whether you can determine if the function $k : \mathbb{R}^2 \to \mathbb{R}$ could be a valid covariance function, given only that $k(1, 1) = k(2, 2) = 1$ and $k(1, 2) = k(2, 1) = 2$.

[3 marks]

(d) Suppose $k : [0, 1]^2 \to \mathbb{R}$ is a function for which Mercer's condition holds. Show that, for any $x_1, \ldots, x_n \in [0, 1]$ we have: the matrix $\mathbf{C} = ((k(x_i, x_j)))_{i,j=1,\ldots,n}$ is positive definite.
HINT: Consider using properties of the Dirac delta function, δ.

[4 marks]