Paper C24	Setter's name: Mark Cannon	Page 1/2
Lecture title: Dynamical Systems		
Question no: 1		

To be handwritten. Please include question sections (e.g. a, b, c) and the marks attributed to each section.

(a). USING $\displaystyle\lim_{x_1, r \to 0} \frac{x_1}{\ln r} = \lim_{x_2, r \to 0} \frac{x_2}{\ln r} = 0$

WE GET $\dot{x}_1 = \dot{x}_2 = 0$ WHEN $x_1 = x_2 = 0$, SO THE ORIGIN IS AN EQUILIBRIUM POINT.

FOR $0 < r < 1$, $\ln r$ IS NEGATIVE (AND FINITE) AND FOR AN EQM POINT WE REQUIRE

$$\left.\begin{array}{l}\dot{x}_1 = 0\\ \dot{x}_2 = 0\end{array}\right\} \Rightarrow \ln r\, x_1 = -x_2 = -x_1/\ln r \Rightarrow x_1\left[(\ln r)^2 + 1\right] = 0$$

SO THERE ARE NO EQUILIBRIUM POINTS WITH $0 < r < 1$.

[4 marks]

(b). LINEARIZE AROUND $(x_1, x_2) = (0,0)$ USING $\dfrac{\partial \ln r}{\partial x_1} = \dfrac{1}{r} \cdot \dfrac{x_1}{r} = \dfrac{x_1}{r^2}$

AND $\dfrac{\partial}{\partial x_1}(x_1/\ln r) = \dfrac{\ln r - x_1^2/r^2}{(\ln r)^2}$, $\dfrac{\partial}{\partial x_1}(x_2/\ln r) = -\dfrac{x_2}{(\ln r)^2} \cdot \dfrac{x_1}{r^2}$:

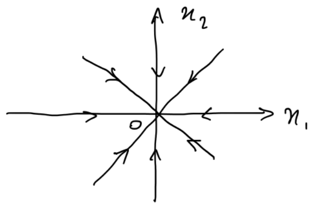
$$f(x_1, x_2) = \begin{bmatrix} -x_1 - x_2/\ln r \\ -x_2 + x_1/\ln r \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} -1 + \dfrac{x_1 x_2}{r^2(\ln r)^2} & -\dfrac{1}{\ln r} + \dfrac{x_2^2}{r^2(\ln r)^2} \\ \dfrac{1}{\ln r} - \dfrac{x_1^2}{r^2(\ln r)^2} & -1 - \dfrac{x_1 x_2}{r^2(\ln r)^2} \end{bmatrix}$$

BUT $\displaystyle\lim_{\substack{r \to 0\\ x_1, x_2 \to 0}} \frac{x_1 x_2}{r^2(\ln r)^2} = \lim_{\substack{r \to 0\\ x_1, x_2 \to 0}} \frac{x_1^2}{r^2(\ln r)^2} = \lim_{\substack{r \to 0\\ x_1, x_2 \to 0}} \frac{r^2}{r^2(\ln r)^2} = 0$, SO

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

EIGENVALUES: $-1, -1 \Rightarrow$ STABLE NODE

PHASE PORTRAIT:



[6 marks]

(c). $r^2 = x_1^2 + x_2^2 \Rightarrow 2 r \dot{r} = 2 x_1 \dot{x}_1 + 2 x_2 \dot{x}_2$

$\tan(\theta) = x_2/x_1 \Rightarrow \sec^2(\theta)\, \dot{\theta} = (x_1 \dot{x}_2 - x_2 \dot{x}_1)/x_1^2$

$\therefore x_1^2\left[1 + (x_2/x_1)^2\right] \dot{\theta} = r^2 \dot{\theta} = x_1 \dot{x}_2 - x_2 \dot{x}_1$

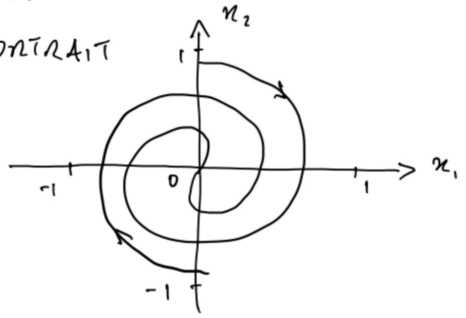
[2 marks]

(d).

$$r\dot{r} = -x_1^2 - \frac{x_1 x_2}{\ln r} - x_2^2 + \frac{x_1 x_2}{\ln r} \Rightarrow \dot{r} = -r$$

$$r^2 \dot{\theta} = -x_1 x_2 + \frac{x_1^2}{\ln r} + x_1 x_2 + \frac{x_2^2}{\ln r} \Rightarrow \dot{\theta} = \frac{1}{\ln r}$$

PHASE PORTRAIT



NEAR $r=0$ WE HAVE

$\dot{r} = -r\,,\ \dot{\theta} \simeq 0$

WHICH AGREES WITH THE LINEARIZATION

[4 marks]

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(a). DIFFERENTIATING $V(x_1, x_2)$ W.R.T. TIME t:

$\dot{V} = x_2 \dot{x}_2 + g(x_1)\dot{x}_1$

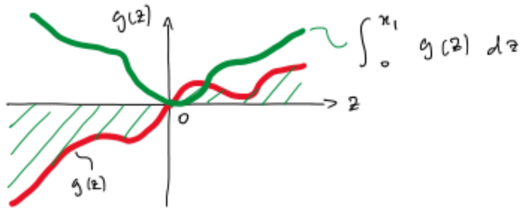
$= -x_2 g(x_1) + x_2 g(x_1) = 0$

THEREFORE $V(x_1, x_2)$ REMAINS CONSTANT ALONG ALL SOLUTION TRAJECTORIES

[4 marks]

(b). – SINCE $g(z) > 0 \ \forall z > 0$ AND $g(z) < 0 \ \forall z < 0$:

$\int_0^{x_1} g(z)\, dz$ INCREASES AS $|x_1|$ INCREASES:



THEREFORE THE MINIMUM OF $V(x_1, x_2)$ OCCURS AT $(x_1, x_2) = (0, 0)$ AND THE CONTOURS $\{(x_1, x_2) : V(x_1, x_2) = c\}$ ARE CLOSED CURVES ENCIRCLING $(0,0)$ FOR SUFFICIENTLY SMALL $c > 0$.

– WE ALSO KNOW THAT

$\int_0^{x_1} g(z)\, dz \to \infty$ AS $|x_1| \to \infty$

AND IT FOLLOWS THAT, FOR ALL $c > 0$, THE CONTOURS $\{(x_1, x_2) : V(x_1, x_2) = c\}$ ARE CLOSED CURVES.

– SINCE $\dot{V}(x_1, x_2) = 0$ ALONG THE SYSTEM TRAJECTORIES, THEY MUST BE PERIODIC [WITH FINITE PERIOD SINCE g IS CONTINUOUS AND $z\, g(z) > 0 \ \forall z \neq 0$ IMPLIES THAT THE ONLY EQUILIBRIUM POINT IS AT $(0,0)$ AND THAT $(0,0)$ LIES IN THE INTERIOR OF $\{(x_1, x_2) : V(x_1, x_2) \leq c\}$ FOR ALL $c > 0$.]

[6 marks]

(c). SHOW THAT THERE ARE NO PERIODIC SOLUTIONS USING EITHER STABILITY ANALYSIS:

WITH $\dot{x}_2 = -g(x_1) + \varepsilon x_2$ WE HAVE $\dot{V}(x_1, x_2) = \varepsilon x_2^2$

– FOR $\varepsilon < 0$ THERE ARE NO PERIODIC SOLUTIONS

PROVE THIS BY SHOWING THAT EVERY SOLUTION CONVERGES ASYMPTOTICALLY TO $(0,0)$ USING LASALLE'S INVARIANCE PRINCIPLE:

THE SET $\{(x_1, x_2) : V(x_1, x_2) \leq c\}$ IS COMPACT AND INVARIANT FOR ALL $c > 0$.

SO $V(x(t), x_1(t)) \leq c$ AT $t = 0$ IMPLIES $V(x_1(t), x_2(t)) \leq c$ FOR ALL $t > 0$.

SINCE $V \geq 0$, IT MUST SATISFY $\dot{V}(x_1, x_2) \to 0$ AS $t \to \infty$.

HENCE $x_2 \to 0$ AND $\dot{x}_2 \to 0$ WHICH IMPLIES $g(x_1) \to 0$ AND HENCE $x_1 \to 0$

THIS ARGUMENT APPLIES TO ALL TRAJECTORIES SINCE $V(x_1, x_2) \to \infty$ AS $\|(x_1, x_2)\| \to \infty$

– FOR $\varepsilon > 0$ THERE ARE NO PERIODIC SOLUTIONS

PROVE THIS, FOR EXAMPLE, BY NOTING THAT THE SYSTEM WITH TIME REVERSED:

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = g(x_1) - \varepsilon x_2$$

CANNOT HAVE PERIODIC SOLUTIONS BECAUSE ITS TRAJECTORIES MUST CONVERGE TO THE ORIGIN, BY THE ARGUMENT USED ABOVE (FOR THE CASE OF $\varepsilon < 0$), SINCE $\dot{V} = -\varepsilon x_2^2 \leq 0$.

<u>OR</u> USING BENDIXSON-DULAC THEOREM:

$$\dot{x}_1 = f_1(x_1, x_2) = x_2$$
$$\dot{x}_2 = f_2(x_1, x_2) = -g(x_1) + \varepsilon x_2$$

$$\therefore \quad \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \varepsilon = \text{CONSTANT}, \quad \varepsilon \neq 0$$

$\therefore \quad \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2}$ HAS THE SAME SIGN FOR ALL $(x_1, x_2) \in \mathbb{R}^2$

$\therefore$ NO PERIODIC SOLUTIONS (SINCE FOR PERIODIC SOLUTIONS IN A REGION D, $\dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2}$ MUST CHANGE SIGN OR BE IDENTICALLY EQUAL TO 0 IN D).

[6 marks]