A10613H1

TRINITY TERM 2025

SECOND PUBLIC EXAMINATION

Honour School of Engineering Science

PROBABILITY, SYSTEMS AND PERTURBATION METHODS (Paper C24)

Monday 16th June 2025 9.30 a.m. – 11.00 a.m.

Answers to not more than **THREE** *questions should be submitted.*

Answer each question in a separate booklet.

DO NOT TURN THIS PAGE UNTIL TOLD YOU MAY DO SO

Note that:

- *The approximate allocation of marks is given in the margin.*
- *Permitted calculator series: Casio fx-83, Casio fx-85, Sharp EL-531.*
- *A copy of Engineering Tables & Data (HLT) is provided.*
- *A cover sheet is provided.*

1. The nonlinear function $f(x;c)$, defined as

$$f(x;c) = \frac{x^2 + c}{2x},$$

in which the variable x is real and parameter c is a real, positive constant, is useful for iterative calculations.

(a) Consider the one-dimensional iterative map $x_n \mapsto x_{n+1}$ in which x_{n+1} is defined as

$$x_{n+1} = f(x_n; c).$$

Identify all fixed points afforded by this discrete mapping. For what computational task do the fixed points suggest that this mapping might be used?

[4 marks]

(b) Given an initial guess of $x_0 = 1/2$, make a table showing the values x_1, x_2, x_3, x_4, and x_5 obtained with the mapping when the parameter $c = 4$, that is, where $x_n \mapsto f(x_n; 4)$. Carefully draw a cobweb plot that illustrates this sequence of calculations, with the axes of your graph covering the domain $0 \le x \le 6$ and the range between 0 and 6.

[5 marks]

(c) What happens if you begin your process of iterating $f(x; 4)$ with the initial guess $x_0 = -1$?

[1 mark]

(d) Rewrite the mapping $x_n \mapsto x_{n+1}$ expressed by $f(x;c)$ in terms of a variable w that expresses the distance from the single fixed point x_+ with the property $x_+ > 0$. Derive the functional form of the mapping $w_n \mapsto w_{n+1}$. Analyze this result to determine a domain of initial values x_0 for which every subsequent iteration of the mapping $x_n \mapsto x_{n+1}$ is guaranteed to get closer to x_+. Hence explain why the domain $x > 0$ is a basin of attraction for x_+.

[6 marks]

2. A nonlinear mechanical oscillator is governed by the first-order autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - 2b \sinh x \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x \tag{1}$$

in which the parameter b is a constant.

(a) Show that the governing system in Equation 1 is equivalent to a second-order ordinary differential equation of the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$

and identify the function $f(x)$. By inspection of this equation describe qualitatively how you expect the displacement $x(t)$ to behave in cases where, for all x, the function $f(x)$ is strictly positive, zero, or strictly negative.

[4 marks]

(b) The system in Equation 1 has a single equilibrium point. Identify the coordinates (x^*, y^*) of this point. Hence use properties derived from a local linearization to establish the values of b for which (x^*, y^*) is a hyperbolic equilibrium of the governing system.

[6 marks]

(c) Explain the basic properties of the function

$$V(x, y) = x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}\left(2b \sinh x - y\right)^2$$

which make it suitable as a Lyapunov function to test the stability of the equilibrium point (x^*, y^*) from part (b). Then use $V(x, y)$ to determine whether (x^*, y^*) is stable when $b > 0$.

[5 marks]

(d) Is the fixed point (x^*, y^*) from part (b) asymptotically stable when $b > 0$?

[1 mark]

3. Consider the following initial-value problem (IVP):

$$\frac{d^2y}{dt^2} + 2\epsilon\frac{dy}{dt} + y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad \left.\frac{dy}{dt}\right|_0 = 1.$$

(a) Use a naive perturbation expansion to derive an approximate solution to this IVP that is valid to leading order in ϵ. Identify the range of t for which your solution is a good approximation and justify your answer.

[5 marks]

(b) Derive the exact solution to the original IVP. By comparison, identify the main problem with your approximate solution from part (a). Explain why the naive approach is unable to capture this feature of the exact solution.

[3 marks]

(c) Use the Method of Multiple Scales to derive a leading-order solution that provides a good approximation for a much larger range of t. Explain why this solution is better than your solution from part (a). Note: Your choice of scales should be informed by your observations in part (b).

[5 marks]

(d) Compare your approximate solution from part (c) with the exact solution from part (b). What are the shortcomings of this approximate solution? Suggest two strategies for improving it.

[3 marks]

4. (a) Suppose the height above ground of the flight trajectory of an unidentified flying object (UFO) is given by the function $h : \mathbb{R} \to \mathbb{R}_+$ where $\mathbb{R}_+$ is the set of positive real numbers, $h(t)$ denotes the height (in m) at time t. Consider the Gaussian process $\mathcal{GP}(\mu, k)$ where the covariance function is given by $k(t, t') = \exp(-4|t - t'|^2)$ and where the mean function is $\mu(t) = 10$ for all times $t, t' \in \mathbb{R}$.

(i) Argue briefly whether or not this Gaussian process might be a reasonable *a priori* model of a subjective (Bayesian) belief about the height trajectory of the UFO.

(ii) Consider *a priori* knowledge about the log-height $\ell(t) = \log(h(t))$, modelled by supposing *a priori* that $\ell \sim \mathcal{GP}(\mu, k)$. Subsequently, the observation $h(\xi) = \exp(6)$ is made for time $\xi = 2$ which is used to create a data set D.
Give the formulae for the posterior mean function $m_{\ell|D}(t) = \mathbb{E}[L_t|D]$ and variance function $v_{\ell|D}(t) = \mathbb{V}[L_t|D]$ for the log-height random variable $L_t = \ell(t)$ at time t. Evaluate them at time $t = 3$ (give an approximate numerical value for each).

(iii) Let H_t denote the random variable of the height $h(t)$ at time t after having observed the data D. Conditional on the data, calculate a 95% confidence interval of the height in the infinite time limit. That is, specify $a, b \in \mathbb{R}$ such that
$\Pr[\lim_{t\to\infty} H_t \in [a, b]] \geq 95\%$.
Hint: Suppose $(X_t)_{t\in\mathbb{R}}$ is a sequence of Gaussian random variables with $X_t \sim \mathcal{N}(m_t, \sigma_t^2)$ for all t and where the moments converge:
$\lim_{t\to\infty} m_t =: m_\infty \in \mathbb{R}, \quad \lim_{t\to\infty} \sigma_t =: \sigma_\infty \in \mathbb{R}$. Then we can define the limit $\lim_{t\to\infty} X_t =: X_\infty$ as a random variable with distribution $\mathcal{N}(m_\infty, \sigma_\infty^2)$.

[9 marks]

(b) Suppose there are two valid continuous covariance functions $k_1, k_2 : \mathbb{X} \to \mathbb{R}$ on the domain $\mathbb{X} = [0, 1]^2 \subset \mathbb{R}^2$. Show that on $\mathbb{X}$ the function $k = k_1 + k_2$ is also a valid covariance function.

[4 marks]

(c) Consider a distribution $P(A, B, C, D, E)$ of the discrete random variables A, B, C, D, E and suppose the distribution has a conditional independence structure admitting the factorisation $P(A, B, C, D) = P(A|B, C)P(C)P(D|C)P(B)P(E)$. Draw a Bayes net representing this conditional independence structure.

[3 marks]