Lecture 1: Introduction

Mark Cannon
mark.cannon@eng.ox.ac.uk

Michaelmas Term 2021

With thanks to Charles Monroe and Antonis Papachristodoulou
for permission to use their course materials from previous years

Lecture 1: Introduction

- In this course, we will study the topological properties of solutions of ordinary differential equations, which can be done without solving them

- Essentially about the geometry of the paths that describe the time evolution of solutions, and how such paths can be thought of as lying on ‘surfaces’

- Intimately associated with the idea of state or phase space and how solutions are related to the ‘state’

Course summary

1. Introduction to dynamical systems
2. Phase space and equilibria
3. The stable, unstable and centre subspaces
4. Lyapunov functions, gradient and Hamiltonian systems; vector fields possessing an integral
5. Invariance. La Salle's theorem. Domain of attraction
6. Limit sets, attractors, orbits, limit cycles, Poincaré maps
7. Saddle-node, transcritical, pitchfork and Hopf bifurcations
8. Logistic map, fractals and Chaos. Lorenz equations

Course structure

- 8 Lectures
- Q&A sessions Thom LR2:
  - L1-4: MT week 7: Fri @ 4pm
  - L5-8: MT week 8: Fri @ 4pm
- Classes Thom LR6:
  - L1-4: MT week 8: Mon @ 10,11,12pm Tue @ 10,11am
  - L5-8: HT week 1: tbc
- Lecture notes, slides & recorded lectures available on Canvas

Example dynamical system: logistic equation

- The logistic equation models population growth of a single species in an environment:

$$\frac{dx}{dt} = bx\Big(1 - \frac{x}{K}\Big)$$

x :	population at time t
$b > 0$:	birth rate
$K > 0$:	carrying capacity of the environment

- Solution (see lecture notes):

$$\left.\begin{matrix} t = \tau/b \\ x = K\theta \end{matrix}\right\} \implies \frac{d\theta}{d\tau} = \theta(1-\theta) \implies \int_{\theta_0}^{\theta} \frac{d\theta}{\theta(1-\theta)} = \int_0^{\tau} d\tau \text{ or } \ln\Big[\frac{\theta(1-\theta_0)}{(1-\theta)\theta_0}\Big] = \tau$$

$$\left.\begin{matrix} \tau = bt \\ \theta = x/K \end{matrix}\right\} \implies x = \frac{Kx_0e^{bt}}{K + x_0(e^{bt} - 1)} \text{ so } x_0 = \frac{cK}{c+K} \implies \boxed{x(t) = \frac{Kce^{bt}}{K + ce^{bt}}}$$

The logistic equation and its solution

The logistic equation

$$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right) \quad \text{has the solution} \quad x(t) = \frac{Kce^{bt}}{K + ce^{bt}}$$

- We have an analytic solution, but what does it say?
  Is it informative to have this exact answer?
- What happens if $x(0) = 0$? and what does this signify?
- What happens as $t \to \infty$? Does $x(t) \to K$?
- Can we analyse these properties without solving the equation?

To address these questions we introduce geometric ideas
into the problem

Phase space

- An nth order ordinary differential equation (ODE) in a single variable $x(t)$ can be written as n coupled differential equations in n variables $x_1(t), x_2(t), \ldots, x_n(t)$

For example $$\frac{d^2x}{dt^2} - b\frac{dx}{dt} + cx = g$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \frac{dx_1}{dt} \end{aligned}\right\} \implies \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

- Each variable $x_1, x_2, \ldots, x_n$ defines a coordinate in phase (or state) space

- Solutions of the ODE create curves (or trajectories) in phase (or state) space, which are determined by the intial conditions

Phase space: terminology

- To emphasise geometric ideas we use concepts from geometry to name phase spaces:
  - if $n = 1$ (1st order ODE) we have a phase line
  - if $n = 2$ (2nd order ODE) we have a phase plane
  - if $n > 2$ we have a general phase space

- We will find that collections of similar trajectories that solve a problem form surfaces, which are sometimes called solution manifolds (a fancy name for a smooth surface)

- Recall that this is not totally new: phase space was mentioned in P1 'Mathematical modelling'!

Returning to the logistic equation example

- The logistic equation: $$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$

- This equation is 1st order ($n = 1$) so the solution trajectories will lie on a phase line

- Also, there are special points where the solution remains stationary (i.e. $x(t)$ does not change with time).
  such stationary points occur if

$$\frac{dx}{dt} = 0 \implies bx\left(1 - \frac{x}{K}\right) = 0 \implies x = 0 \text{ or } K$$

- These special points are also called equilibria (of the equation)

Logistic equation on the phase line

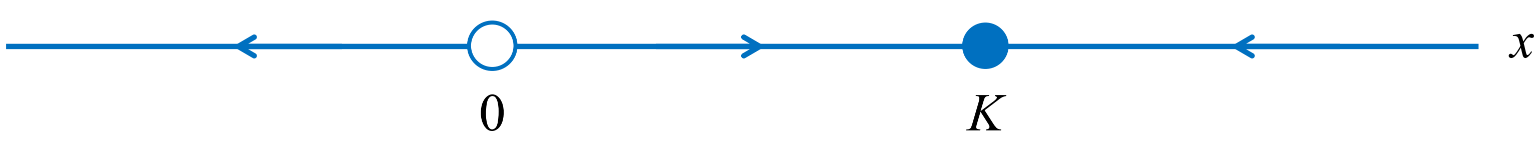
- The logistic equation: $$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$

- Instead of solving this (to find how x varies with t), consider how the rate of change of x depends on x

$$x < 0 \implies \frac{dx}{dt} < 0$$

$$0 < x < K \implies \frac{dx}{dt} > 0$$

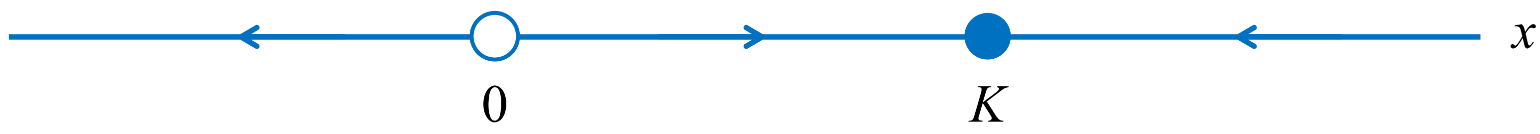
$$x > K \implies \frac{dx}{dt} < 0$$



phase portrait of the logistic equation

Stable and unstable equilibria

- The logistic equation: $$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$



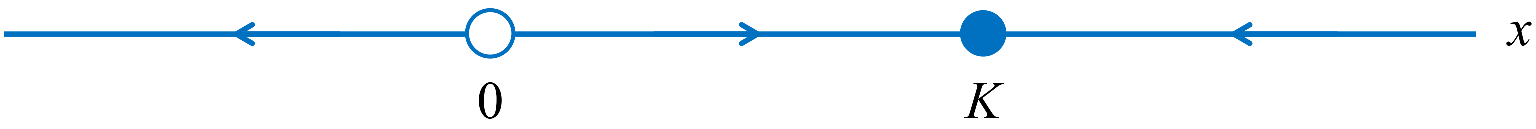
- All trajectories near $x = 0$ move away from that equilibrium point: we call such an equilibrium unstable

- All trajectories near $x = K$ flow toward that equilibrium point: such an equilibrium is called stable

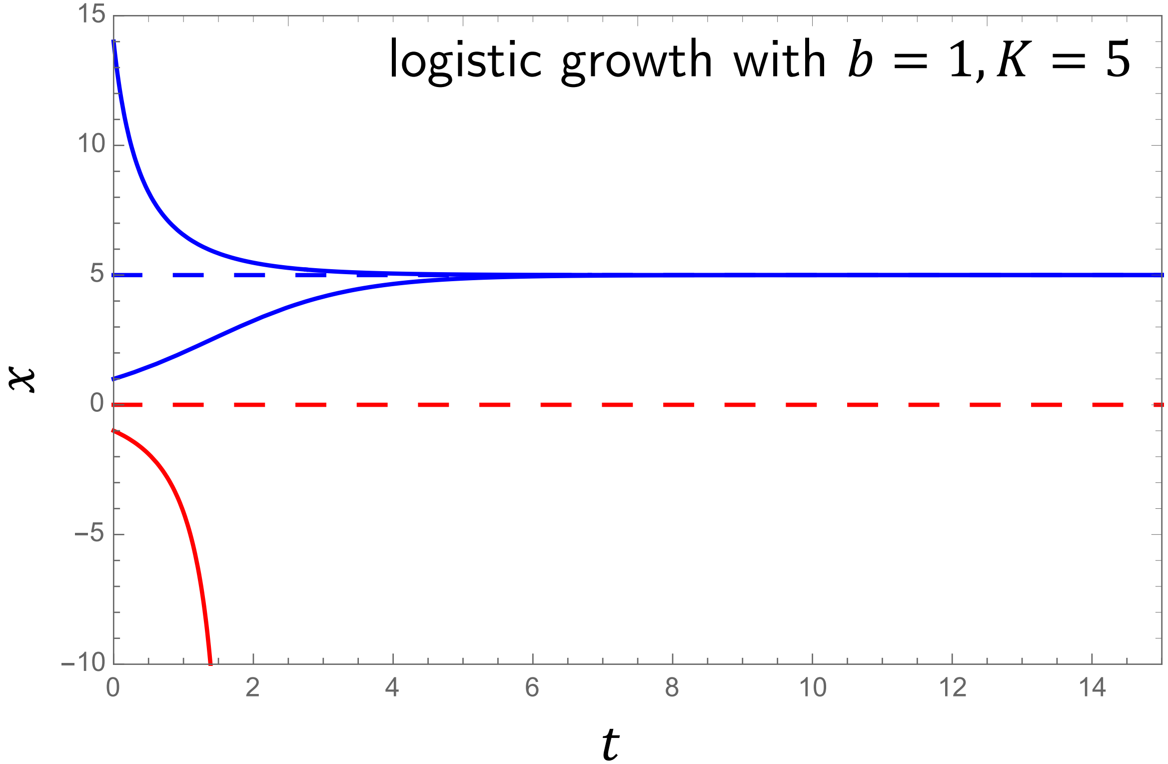
- Note also that it is not possible to move from values $x < K$ to values $x > K$ without crossing $x = K$; this makes motion stop! Thus there is no overshoot at $x = K$

Logistic equation: visualizing solutions

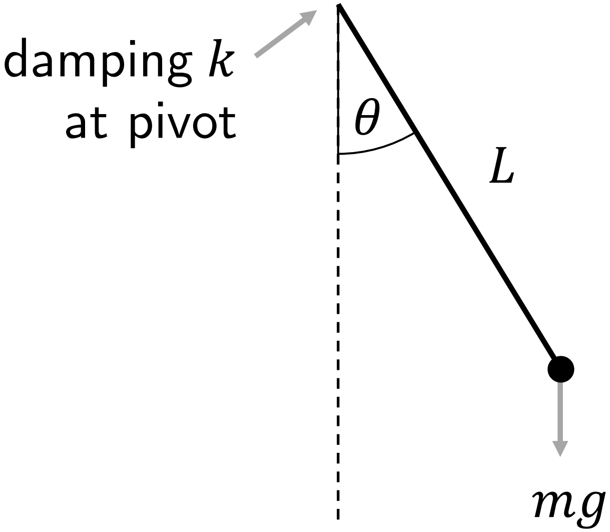
- The logistic equation: $$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$



- Solutions $x(t)$ for a few indicative initial conditions x_0:



Example: damped single pendulum



- $mL\dfrac{d^2\theta}{dt^2} = -mg\sin\theta - kL\dfrac{d\theta}{dt}$
- Define $\tau = t\sqrt{\frac{g}{L}}$, $b = \frac{k}{m}k\sqrt{\frac{L}{g}}$, then

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta - b\frac{d\theta}{d\tau}$$

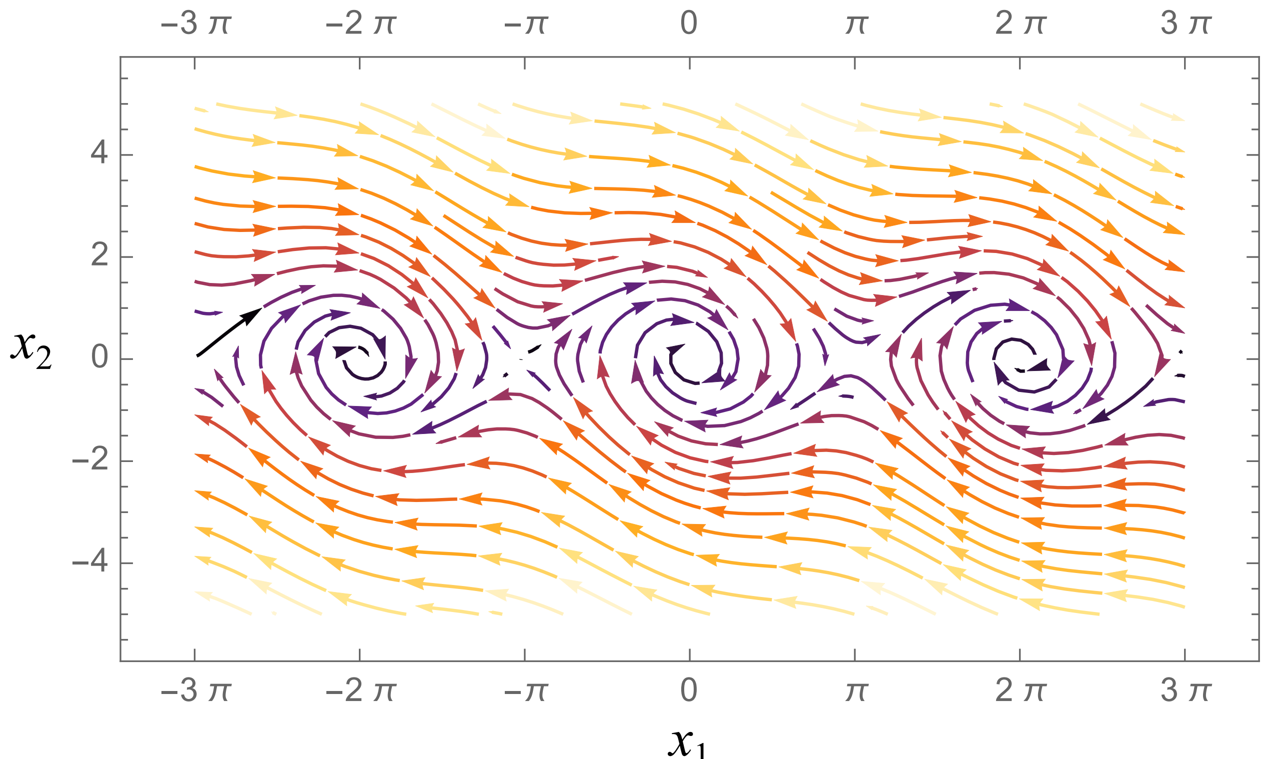
- 2nd order, so two states; let $x_1 = \theta$ and $x_2 = \dfrac{d\theta}{d\tau}$:

$$\frac{dx_1}{d\tau} = x_2$$
$$\frac{dx_2}{d\tau} = -\sin x_1 - bx_2$$

Phase plane for damped pendulum

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta - b\frac{d\theta}{d\tau} \qquad \Longrightarrow \qquad \begin{aligned} \frac{dx_1}{d\tau} &= x_2 \\ \frac{dx_2}{d\tau} &= -\sin x_1 - b x_2 \end{aligned}$$

- Equilibrium points at $x_2 = \sin x_1 = 0$



- Phase plane of the damped pendulum, $b = 1/\sqrt{10}$

Example: glycolitic oscillations

- Glycolysis is a process in which glucose is turned into energy compounds like ATP inside cells

$$\frac{dx}{dt} = -x + ay + x^2 y$$
$$\frac{dy}{dt} = b - ay - x^2 y$$

$x(t)$ and $y(t)$ represent concentrations of reaction intermediates

- Equilibrium conditions: $\begin{cases} x = b \\ y = \dfrac{b}{a+b^2} \end{cases}$ so let $\begin{cases} x_1 = \dfrac{x}{b} \\ x_2 = \dfrac{(a+b^2)}{b} y \end{cases}$

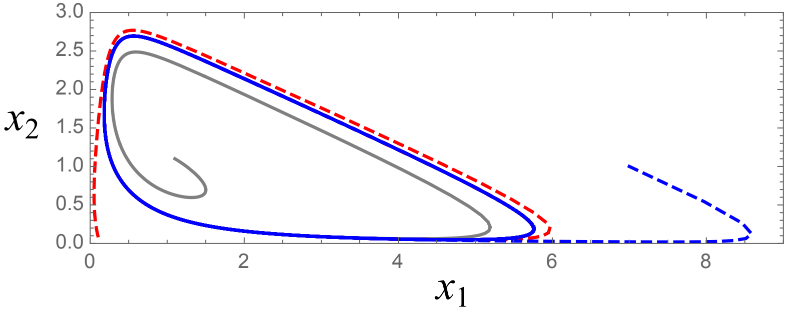
then
$$\frac{dx_1}{dt} = -x_1 + \left(\frac{a}{a+b^2}\right) x_1^2 x_2$$
$$\frac{dx_2}{dt} = (a+b^2)\left[1 - \left(\frac{a}{a+b^2}\right) x_2 - \left(\frac{b^2}{a+b^2}\right) x_1^2 x_2\right]$$

Phase plane for glycolytic oscillations

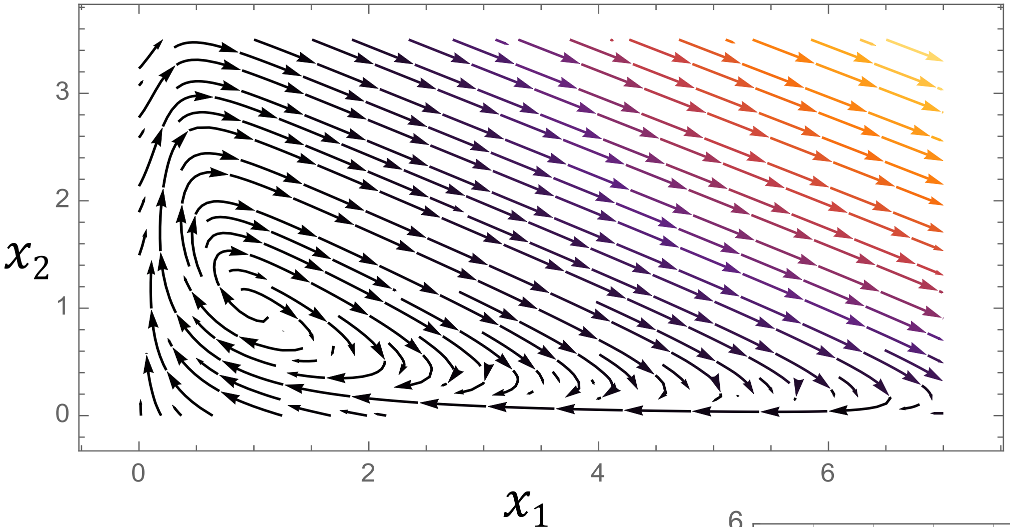
$$\frac{dx_1}{dt} = -x_1 + \frac{a}{a+b^2}x_2 + \frac{b^2}{a+b^2}x_1{}^2 x_2$$

$$\frac{dx_2}{dt} = \left(a+b^2\right)\left[1 - \frac{a}{a+b^2}x_2 - \frac{b^2}{a+b^2}x_1{}^2 x_2\right]$$

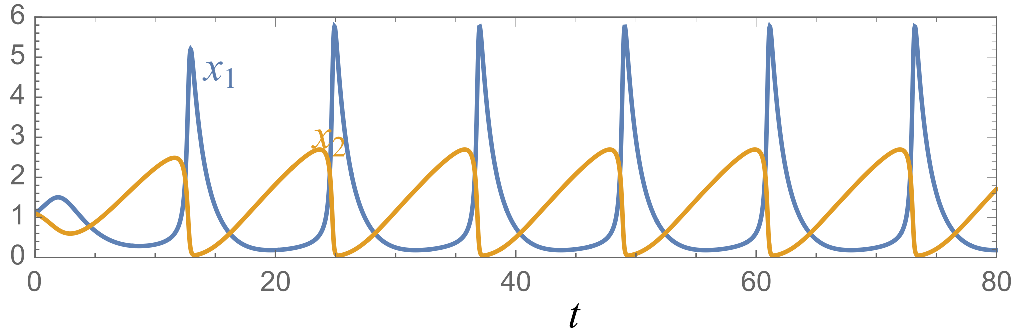
- Study oscillations with $a = 0.03$, $b = 0.6$
- Trajectories for three ICs: $\{x_1(0), x_2(0)\} = \{0.1,0.1\},\{7,1\},\{1.1,1.1\}$



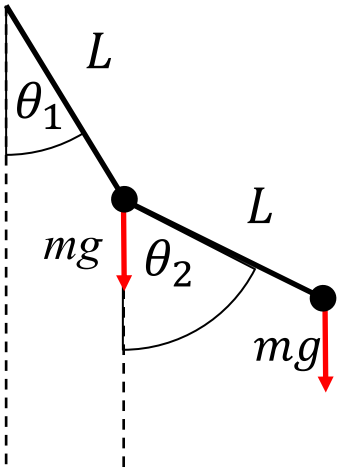
- Phase plane



- Transients for {1.1,1.1}



Example: double pendulum

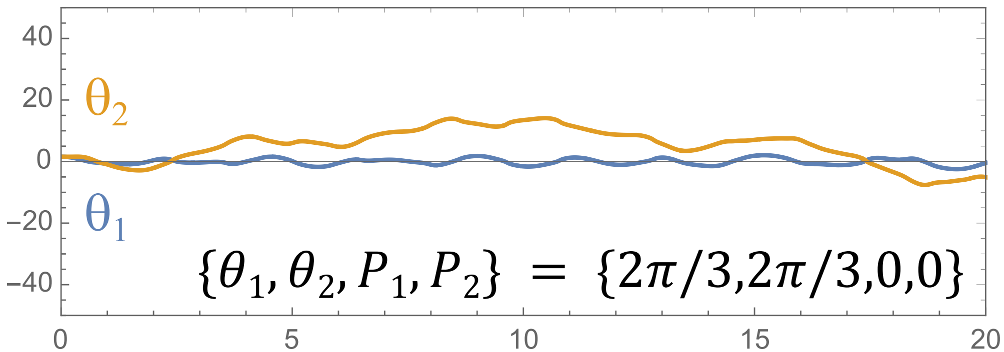


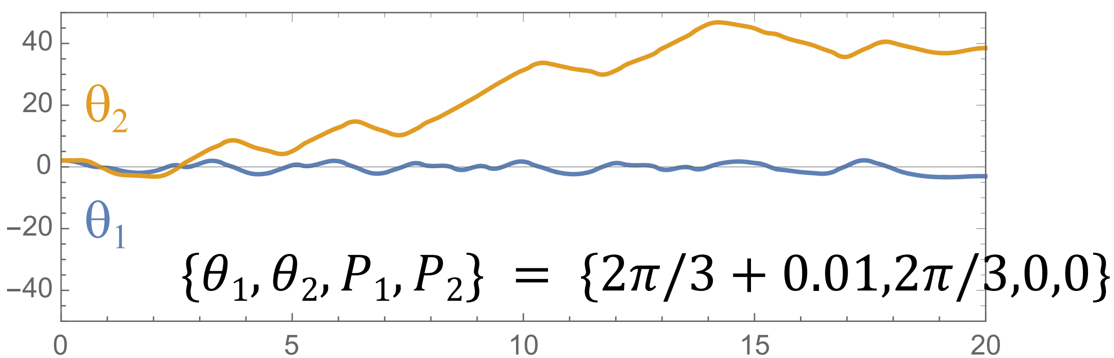
$$\frac{d\theta_1}{dt} = \frac{P_1 - P_2\cos(\theta_1-\theta_2)}{1+\sin^2(\theta_1-\theta_2)}$$

$$\frac{d\theta_2}{dt} = \frac{2P_2 - P_1\cos(\theta_1-\theta_2)}{1+\sin^2(\theta_1-\theta_2)}$$

$$\frac{dP_1}{dt} = -2\frac{g}{L}\sin\theta_1 - \frac{\left[2P_1P_2-\left(P_1^2+2P_2^2\right)\cos(\theta_1-\theta_2)\right]\sin(\theta_1-\theta_2)}{\left[1+\sin^2(\theta_1-\theta_2)\right]^2}$$

$$\frac{dP_2}{dt} = -\frac{g}{L}\sin\theta_2 + \frac{\left[2P_1P_2-\left(P_1^2+2P_2^2\right)\cos(\theta_1-\theta_2)\right]\sin(\theta_1-\theta_2)}{\left[1+\sin^2(\theta_1-\theta_2)\right]^2}$$

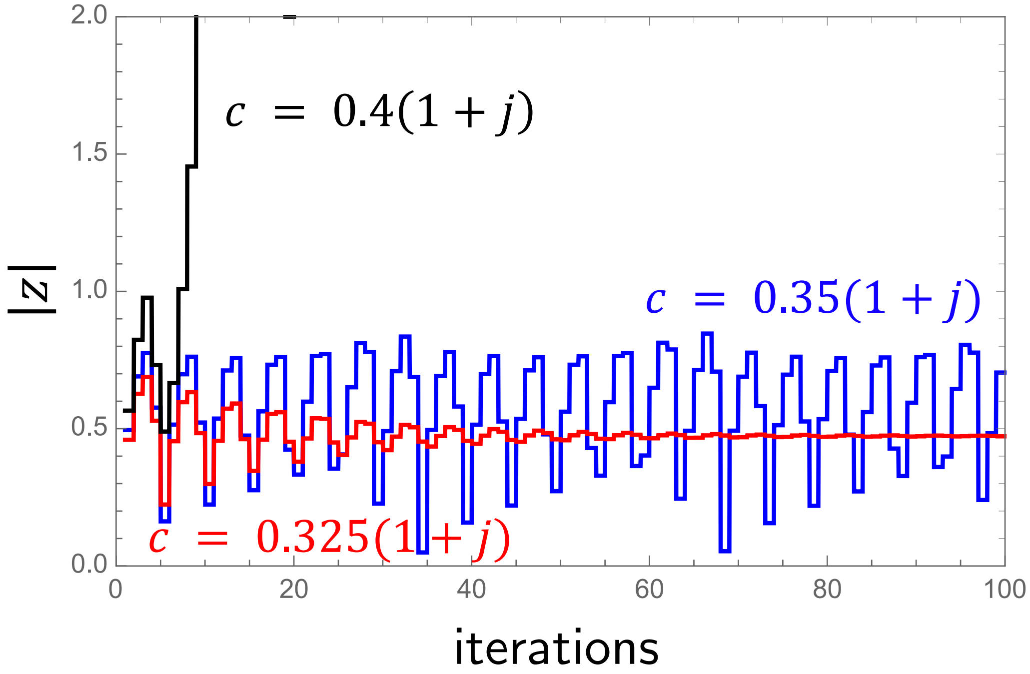




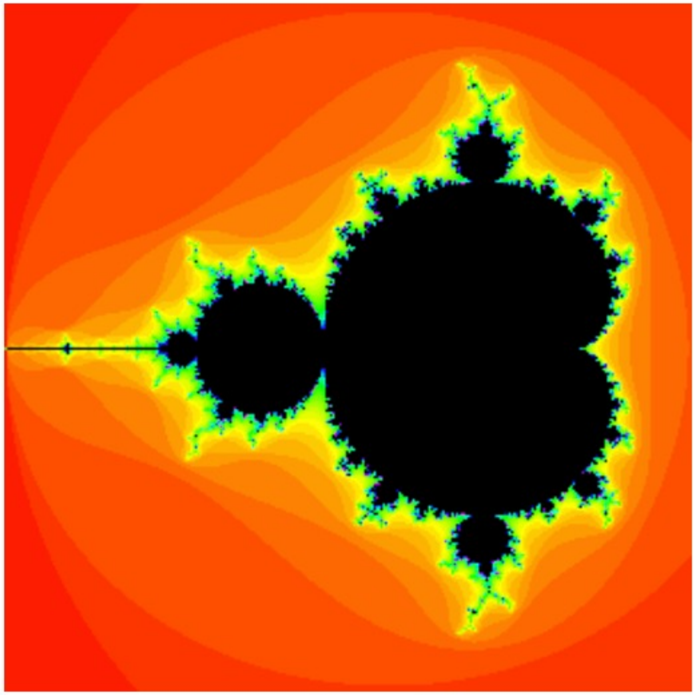
- The two solutions start very near the same point in phase space, but the transients differ dramatically – why?

Example: the Mandelbrot set

- An iterative equation: $z_{k+1} = z_k^2 + c$ with c a complex number
- Question: if $z_0 = 0$, for which values of c does $|z_k|$ remain bounded?
- Effect of varying c values



- The set for all complex c



Our strategy

- We will find equilibrium points of differential equations
- The nature of each equilibrium point is largely established by investigating the local linearization
- We then study the geometry and topology (connectedness) of regions around equilibria in phase space
- We reason about behaviours of flows through these regions
- To implement this strategy we will use topics from linear algebra – eigenvalues and eigenvectors of matrices

Eigenvalues and eigenvectors review

- Suppose $\mathbf{A}$ is square matrix that maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^n$

- An eigenvalue λ and corresponding eigenvector $\mathbf{v}$ of $\mathbf{A}$ satisfy

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

- The set of eigenvalues has multiplicity n and is calculated by finding roots of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

- Recall that complex eigenvalues come in conjugate pairs

- Further recall that if all the eigenvalues are real and distinct then there is a complete (n-dimensional) set of eigenvectors

Eigenvectors as a basis

- If a matrix affords a complete set of n eigenvectors, then they will span the space $\mathbb{R}^n$

- Hence the set of eigenvectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$

- Put another way, any vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written uniquely as a linear combination of the eigenvectors in this case:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \sum_{i=1}^{n} c_i\mathbf{v}_i$$

- Note that if the eigenvalues of a matrix are not distinct (i.e., a root of the characteristic equation repeats) then there may not be a complete set of eigenvectors – there will be a set of 'generalized eigenvectors', however (see Perko Ch. 1)

Matrix diagonalization

- If a real $n \times n$ matrix $\mathbf{A}$ has n distinct real eigenvalues, then there is a complete set of real eigenvectors that span $\mathbb{R}^n$

- In this case $\mathbf{A}$ is diagonalizable, that is, there exists an invertible matrix $\mathbf{V}$ and diagonal matrix $\mathbf{\Lambda}$ such that

$$\mathbf{AV} = \mathbf{V\Lambda} \implies \mathbf{A} = \mathbf{V\Lambda V}^{-1}$$

- The kth diagonal entry of $\mathbf{\Lambda}$ is the kth eigenvalue of $\mathbf{A}$ the kth column of $\mathbf{V}$ is the corresponding eigenvector

Complex eigenvalues

- If a $n \times n$ matrix $\mathbf{A}$ has complex eigenvalues, then its eigenvectors will be complex so $\mathbf{A}$ cannot be diagonalized using matrices of real numbers

- There is a simple way to rearrange the diagonalization in the case of 2×2 matrices with complex eigenvalues:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad \Longrightarrow \quad \begin{array}{l} \lambda \in \{a + jb,\ a - jb\} \\ \mathbf{v} \in \{\mathbf{u} + j\mathbf{w},\ \mathbf{u} - j\mathbf{w}\} \end{array}$$

- Let $\mathbf{V}' = [\mathbf{w} \quad \mathbf{u}]$, then we can write the standard form

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \mathbf{V}' \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{V}'^{-1}$$

Example of complex diagonalization

- Consider $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ with eigenvalues: $\lambda_1 = 2 + j$, $\lambda_2 = 2 - j$

- eigenvectors are solutions of: $\begin{bmatrix} 3-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\implies \quad \mathbf{v}_1 = \begin{bmatrix} 1+j \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1-j \\ 1 \end{bmatrix}$$

- so $\lambda = a \pm jb$, $\mathbf{v} = \mathbf{u} \pm j\mathbf{w}$ with $(a, b) = (2, 1)$, $(\mathbf{u}, \mathbf{w}) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

$$\mathbf{A} = \begin{bmatrix} 1+j & 1-j \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2+j & 0 \\ 0 & 2-j \end{bmatrix} \begin{bmatrix} 1+j & 1-j \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

Linear autonomous systems

- An autonomous system of first-order differential equations depends on the dependent variables, but does not explicitly include the independent variable:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \qquad\qquad \frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

autonomous   not autonomous

- The linear autonomous system of first-order differential equations can be written in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

- Define $e^{\mathbf{A}} \triangleq \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$ then if $\mathbf{A}$ is diagonalizable

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$$

Computation of matrix exponentials

- To compute the matrix exponential, we use diagonalization:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)^k}{k!} = \mathbf{V}\Big(\sum_{k=0}^{\infty} \frac{\mathbf{\Lambda}^k}{k!}\Big)\mathbf{V}^{-1} = \mathbf{V}\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}\mathbf{V}^{-1}$$

- This is straightforward if the eigenvalues are real and distinct

- For complex eigenvalues, we can use the standard form of a 2×2:

since $\qquad \lambda = a \pm jb \quad \Longrightarrow \quad e^{\lambda} = e^{a}(\cos b \pm j \sin b)$

we get $\qquad e^{\mathbf{A}} = \mathbf{V}' \begin{bmatrix} e^{a}\cos b & -e^{a}\sin b \\ e^{a}\sin b & e^{a}\cos b \end{bmatrix} \mathbf{V}'^{-1}$

- See Perko *Differential equations and dynamical systems*, sec 1.5

Definitions for dynamical systems 1

- Generally, ordinary differential equations of order n can be represented instead as a set of n coupled 1st-order ODEs

- Each of the n dependent variables in this ODE system is called a **state**, $x_i(t) \in \mathbb{R}$, and $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector

- Generally we can write the ODE system as a list of functions $\dot{x}_i = f_i(\mathbf{x})$ where f_i is a mapping from the vector space of states into the real numbers, i.e. $f_i : \mathbb{R}^n \to \mathbb{R}$

- If f_i is defined on a subspace $D \subset \mathbb{R}^n$, called the **domain** of the function, then we write $f_i : D \to \mathbb{R}$
(e.g. the square root function is limited to $D = \{x : x \geq 0\}$)

- We let $\mathbf{f}$ represent the vector whose ith entry is f_i, so $\mathbf{f} : D \to \mathbb{R}^n$

Definitions for dynamical systems 2

- The general set of nonlinear **autonomous** systems can be written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

- Systems can have **parameters** so $\mathbf{f}$ may depend on a vector $\mathbf{p} \in \mathbb{R}^p$:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; \mathbf{p}) \implies \frac{d\mathbf{x}}{dt} \text{ is a function } \mathbf{f} \text{ of } \mathbf{x} \text{ parameterised by } \mathbf{p}$$

- We can also consider **difference equations** (also called maps), which are not ODEs but follow recurrence relations:

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k; \mathbf{p})$$

Bird's-eye view of dynamical systems

- Given an autonomous ordinary differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x};\mathbf{p})$$

- A solution of the ODE is a map from the time interval $t \in \{\alpha, \beta\}$ into the space $\mathbb{R}^n$, which passes through initial condition $\mathbf{x_0} \in \mathbb{R}^n$

$$\mathbf{x} : (\alpha, \beta) \to \mathbb{R}^n \text{ such that } \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t);\mathbf{p}) \text{ and } \mathbf{x}(0) = \mathbf{x}_0$$

- Note that if the initial condition is at $t = 0$, then $\alpha < 0$ and $\beta > 0$

- Here we will not be concerned with solving such equations – instead we will look at the geometry of these solutions

Existence and uniqueness of solutions

- Does a solution **exist?** Is it **unique?**

- The study of existence and uniqueness is quite technical...

- The lecture notes describe an aside (non-examinable!) considering existence (Lipschitz continuity), see
  - lecture notes sec. 1.3.1
  - Perko sec. 2.2 & 2.3