Lecture 2: Equilibria and Stability

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- Equilibrium definitions
- Stability definitions for equilibrium points
- Phase space of Linear systems
- Linearization and the stability of equilibria of nonlinear systems

Equilibria of continuous time systems

An **equilibrium** is a point in state space where $\dot{\mathbf{x}} = 0$:

$$\boxed{\mathbf{x}^* \text{ is an equilibrium of } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ if and only if } \mathbf{f}(\mathbf{x}^*) = 0}$$

- If $\mathbf{x}(0) = \mathbf{x}^*$, then $\mathbf{x}(t) = \mathbf{x}^*$ for all t
  hence $\mathbf{x}^*$ is sometimes called a **fixed point**

- For a **linear** autonomous system with non-zero eigenvalues, there is only one solution to $A\mathbf{x}^* = 0$, namely $\mathbf{x}^* = 0$

- In general there may be many points $\mathbf{x}^*$ satisfying $\mathbf{f}(\mathbf{x}^*) = 0$
  therefore a **nonlinear** system can have many equilibria

Equilibria of maps

Discrete time systems also have equilibrium points

$\boxed{\mathbf{x}^* \text{ is an equilibrium of } \mathbf{x}_{k+1} = g(\mathbf{x}_k) \text{ if and only if } g(\mathbf{x}^*) = \mathbf{x}^*}$

- The equilibria of a discrete time system are the **fixed points** of the state update equation, so that

$$\mathbf{x}_0 = \mathbf{x}^* \quad \Longrightarrow \quad \mathbf{x}_k = \mathbf{x}^* \text{ for all } k$$

- For differential equations, there is a flow of solutions through phase space but the state of a discrete time system 'jumps' between points space, making their trajectories harder to visualize

Flows and equilibria

- We can think of the solution to a non-linear differential equation as a flow in an n-dimensional phase space, representing it with streamlines as we would for flow of a liquid

- In this analogy the vector-valued function $\mathbf{f}$ in

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

  is a vector field defining the **flow velocity**

- Flows can end or begin at equilibria, or circulate around them

- The stability of the flow near an equilibrium is an important characteristic, which we will focus on today

Stability of flow equilibria

- **Definition**: An equilibrium point $\mathbf{x}^*$ is said to be **stable** if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that all solutions $\mathbf{x}(t)$ satisfy

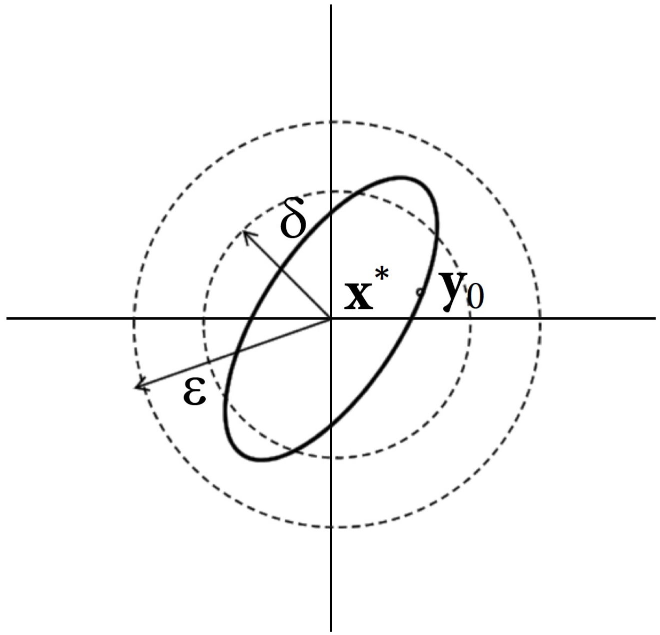
  $$|\mathbf{x}(t) - \mathbf{x}^*| < \varepsilon \text{ for all } t \geq 0 \text{ whenever } |\mathbf{x}(0) - \mathbf{x}^*| < \delta$$

- Otherwise the equilibrium point is said to be **unstable** (i.e. if, for some $\varepsilon > 0$, no $\delta > 0$ exists satisfying this condition)

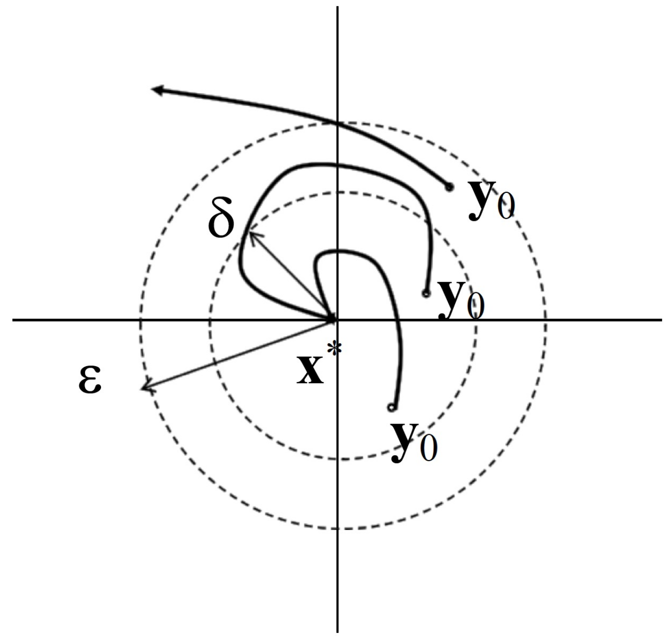
- **Definition**: An equilibrium point $\mathbf{x}^*$ is asymptotically stable if it is stable and $\beta > 0$ exists such that

  $$\lim_{t \to \infty} |\mathbf{x}(t) - \mathbf{x}^*| = 0 \text{ whenever } |\mathbf{x}(0) - \mathbf{x}^*| < \beta$$

Picturing stability



Stable equilibrium



Asympotically stable equilibrium

- The solution cannot escape from a stable equilibrium
- The solution converges to the equilibrium point if it starts close enough to an asymptotically stable equilibrium

Epsilon-delta arguments

- Observe that the argument made within a definition like

$$|\mathbf{x}(t) - \mathbf{x}^*| < \varepsilon \text{ for all } t \geq 0 \text{ whenever } |\mathbf{x}(0) - \mathbf{x}^*| < \delta$$

takes the form of a game:

1. I give you a positive number ε that I am free to choose
2. you respond with a number δ that satisfies some condition
3. if you can find a number δ for any ε, you 'win'

- Many proofs and definitions in mathematics are based on this kind of argument

Exponential stability

- **Definition**: An equilibrium point $\mathbf{x}^*$ is **exponentially stable** if $\mathbf{x}^*$ is asymptotically stable and there exist finite constants α, β, $\delta > 0$ such that

  $$|\mathbf{x}(t) - \mathbf{x}^*| < \alpha e^{-\beta t}|\mathbf{x}(0) - \mathbf{x}^*| \ \forall t \geq 0 \text{ whenever } |\mathbf{x}(0) - \mathbf{x}^*| < \delta$$

- As well as requiring that the solution is stable and converges to the equilibrium point (asymptotic stability), this also quantifies the **rate of convergence**

  i.e. how fast the solution flows to the equilibrium point

Flows in 2x2 linear systems

- Ultimately we will study flows around the equilibrium points of nonlinear ODE systems by examining **local linearizations** about those points

- Each flow has a **topology** (a shape) that falls into one of a number of distinct categories

- The flows in local linearizations can often be continuously distorted into flows that solve the non-linear ODE systems

- It is useful to study the topologies of some example linear systems to understand how families of solutions look

The uncoupled 2x2 first-order linear system

- Perhaps the simplest problem we can think of is

$$\frac{dx_1}{dt} = \alpha_1 x_1 \qquad \text{solved by} \qquad x_1(t) = x_1(0)e^{\alpha_1 t}$$

$$\frac{dx_2}{dt} = \alpha_2 x_2 \qquad\qquad\qquad x_2(t) = x_2(0)e^{\alpha_2 t}$$

- These can be viewed as parametric equations to describe the shapes of curves in phase space

$$\left[x_1(t)\right]^{\alpha_2/\alpha_1} = \left[x_1(0)\right]^{\alpha_2/\alpha_1} e^{\alpha_2 t} = x_2(0)^{-1}\left[x_1(0)\right]^{\alpha_2/\alpha_1} x_2(t)$$

$$x_2 = c{x_1}^{\alpha_2/\alpha_1}$$

Stability of the uncoupled 2x2 system

- Let us inspect the system

$$\frac{dx_1}{dt} = \alpha_1 x_1 \qquad \text{solved by} \qquad x_1(t) = x_1(0)e^{\alpha_1 t}$$

$$\frac{dx_2}{dt} = \alpha_2 x_2 \qquad\qquad\qquad x_2(t) = x_2(0)e^{\alpha_2 t}$$

- The system has an equilibrium point at the origin (it is a linear autonomous system)

- If $\alpha_1 < 0$ and $\alpha_2 < 0$ then the system is asymptotically stable, in fact **exponentially stable**

- If $\alpha_1 > 0$ or $\alpha_2 > 0$ then the origin is **unstable**

Coupled 2x2 first-order linear system

- Complexity goes up if we consider a coupled linear system,

$$\frac{dx_1}{dt} = ax_1 + bx_2 \qquad \text{or} \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a & d \\ c & b \end{bmatrix}$$

$$\frac{dx_2}{dt} = cx_1 + dx_2$$

- Here the solution for initial condition $\mathbf{x}(0)$ will be $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$

- Since the matrix exponential is involved here, we know that if the coefficient matrix is diagonalizable, then

$$e^{t\mathbf{A}} = \mathbf{V}\mathrm{diag}\left\{e^{\lambda_i t}\right\}\mathbf{V}^{-1}$$

and can be constructed using eigenvalues and eigenvectors

Eigenvalues

- Here the eigenvalues of $\mathbf{A}$ are found by solving

$$\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=\det\left(\begin{bmatrix} a-\lambda & d \\ c & b-\lambda \end{bmatrix}\right)=\lambda^2-\left(a+b\right)\lambda+\left(ab-cd\right)=0$$

- This characteristic equation can also be written in terms of the **trace** and **determinant** of $\mathbf{A}$:

$$\lambda^2-\operatorname{tr}\left(\mathbf{A}\right)\lambda+\det\left(\mathbf{A}\right)=0$$

- Generally the trace of a matrix is the sum of its eigenvalues:

$$\operatorname{tr}\left(\mathbf{A}\right)=\lambda_1+\lambda_2$$

- The determinant of a matrix is the product of its eigenvalues:

$$\det\left(\mathbf{A}\right)=\lambda_1\lambda_2$$

Solving with eigenvectors and eigenvalues

- If the eigenvalues of $\mathbf{A}$ are real and distinct, then we can write any initial condition as a combination of the eigenvectors:

$$\mathbf{x}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{V}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{V}^{-1}\mathbf{x}(0)$$

- Then using $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$, we have

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{V}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{V}\mathrm{diag}\left\{e^{\lambda_i t}\right\}\mathbf{V}^{-1}\mathbf{V}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

- The solution is a linear combination of exponential transients with decay rates determined by the eigenvalues

Characteristics of linear system trajectories

- If $\mathrm{Re}(\lambda) < 0$, then the component along the corresponding eigenvector decays to zero
- If $\mathrm{Re}(\lambda) > 0$, then the component along the corresponding eigenvector grows without bound
- If $\lambda = 0$, the component along the corresponding eigenvector remains constant
- If $\mathrm{Im}(\lambda) \neq 0$, then the solution orbits or spirals around the origin
- If $\mathrm{Im}(\lambda) = 0$, then the solution does not orbit or spiral
- If λ is real, the solution tends toward the eigenvector with the dominant eigenvalue

Coordinate transformation to normal form

- Characteristic trajectory shapes in 2-D phase space are found via a coordinate transformation that puts $\mathbf{A}$ in a standard form:
  - Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, define new coordinates $\mathbf{y} = \mathbf{K}^{-1}\mathbf{x}$
  - This transforms the equations of motion to $\dot{\mathbf{y}} = \mathbf{K}\mathbf{A}\mathbf{K}^{-1}\mathbf{y}$
  - If eigenvalues of $\mathbf{A}$ are real and distinct let $\mathbf{K} = \mathbf{V}$: $\mathbf{V}\mathbf{A}\mathbf{V}^{-1} = \mathbf{D}$

$$\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{y}$$

  - If eigenvalues of $\mathbf{A}$ are complex, $\lambda = a \pm jb$, then let $\mathbf{K} = \mathbf{V}'$:

$$\dot{\mathbf{y}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{y}$$

- The transformed matrix is called the normal form of $\mathbf{A}$

Degeneracy of eigenvectors

- In the case that eigenvalues are real but not distinct, i.e. if $\lambda_1 = \lambda_2$, then there may only be a single (degenerate) eigenvector. Example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ has eigenvalue } \lambda = 1 \text{ (multiplicity 2), eigenvector } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- In this case, form a generalized eigenvector, $\mathbf{v}_2$, such that

$$\left(\mathbf{A} - \lambda_1 \mathbf{I}\right)\mathbf{v}_2 = \mathbf{v}_1 \quad \Rightarrow \quad \left(\mathbf{A} - \lambda_1 \mathbf{I}\right)^2 \mathbf{v}_2 = \left(\mathbf{A} - \lambda_1 \mathbf{I}\right)\mathbf{v}_1 = \mathbf{o}$$

- The transformation $\mathbf{V}' = [\mathbf{v}_1 \quad \mathbf{v}_2]$ then expresses $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{V}'\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\left(\mathbf{V}'\right)^{-1}$$

- This is called the **normal form** of a **degenerate** matrix $\mathbf{A}$

Saddle equilibrium points via normal forms

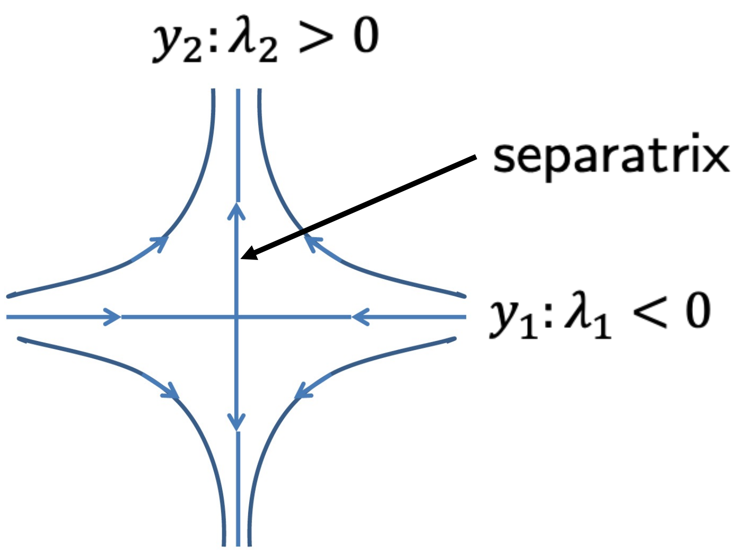
- If the eigenvalues of $\mathbf{A}$ are real and $\lambda_1\lambda_2 < 0$, then the equilibrium of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is unstable and is called a saddle point

Transforming coordinates: $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

- This solution will have four asymptotes that approach the origin, two as $t \to \infty$ and two as $t \to -\infty$

- These four trajectories are called separatrices

a saddle shape in the phase plane:



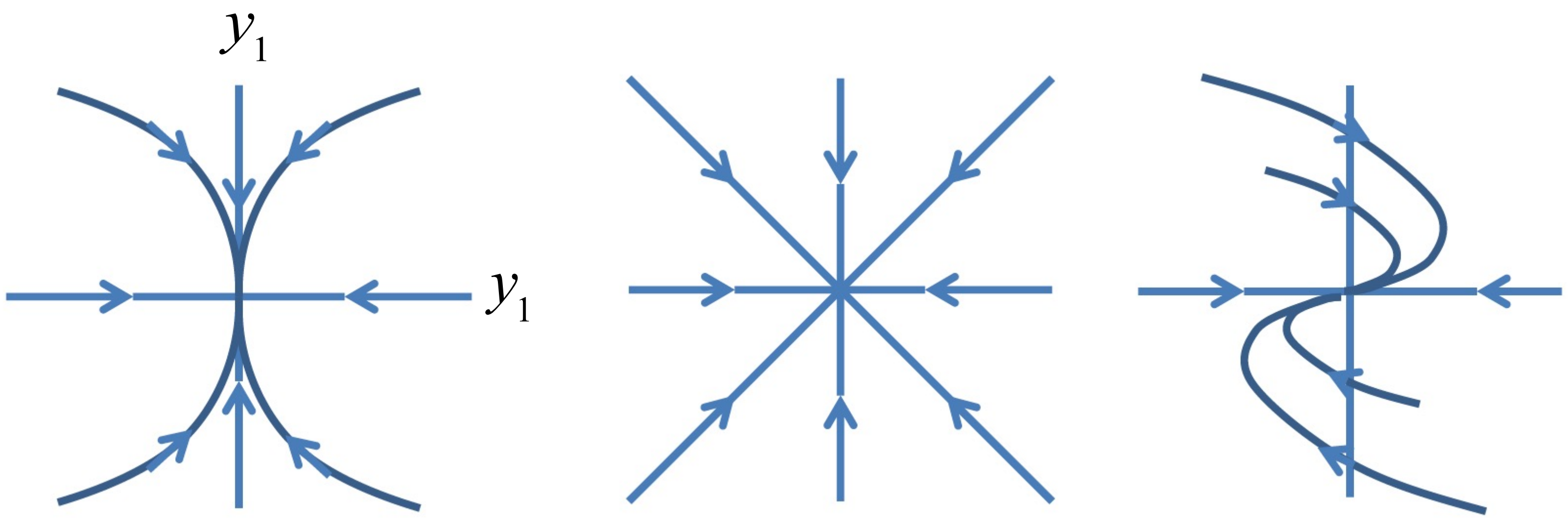
Stable equilibrium points via normal forms

- If the eigenvalues of $\mathbf{A}$ are real and both $\lambda_1 < 0$ and $\lambda_2 < 0$, then the equilibrium will be stable

- Three cases:
  - Eigenvalues distinct, $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
  - Eigenvalues repeated but two eigenvectors: $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$
  - Eigenvalues repeat but $\mathbf{A}$ degenerate: $(\mathbf{V}')^{-1}\mathbf{A}(\mathbf{V}') = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$

Stable and unstable flow shapes

- Consider the transformed coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$ or $\mathbf{y} = (\mathbf{V}')^{-1}\mathbf{x}$
- Stable solutions when both eigenvalues are negative:



standard case, $\lambda_1 < \lambda_2$ nondegenerate, $\lambda_1 = \lambda_2$ degenerate

- If both eigenvalues are positive then the solutions are unstable; flows in the phase plane look the same as above, but the arrows point in the opposite directions

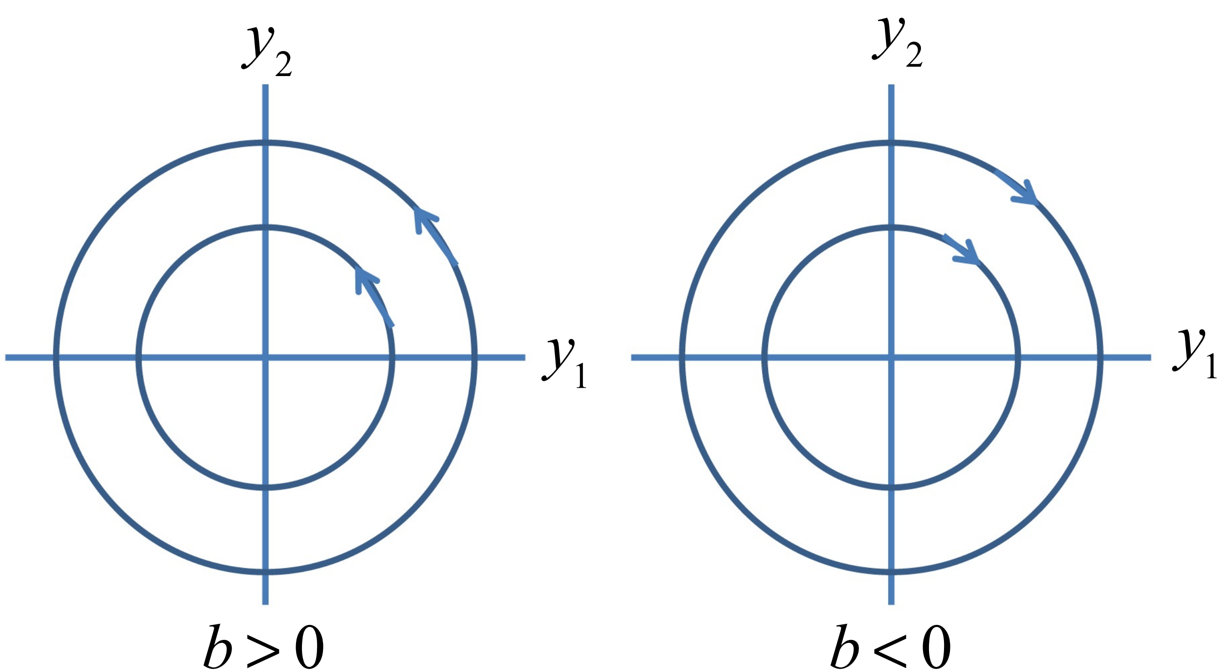
Centre equilibrium points via normal forms

- If the eigenvalues are purely imaginary (real part equal to zero), then the corresponding equilibrium point is marginally stable

- In this case the form from slide 16 (with transformation from lecture 1, slide 23) shows that the normal form is

$$\mathbf{V}'\mathbf{A}\left(\mathbf{V}'\right)^{-1} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

- Phase portraits:



Stable and unstable spirals

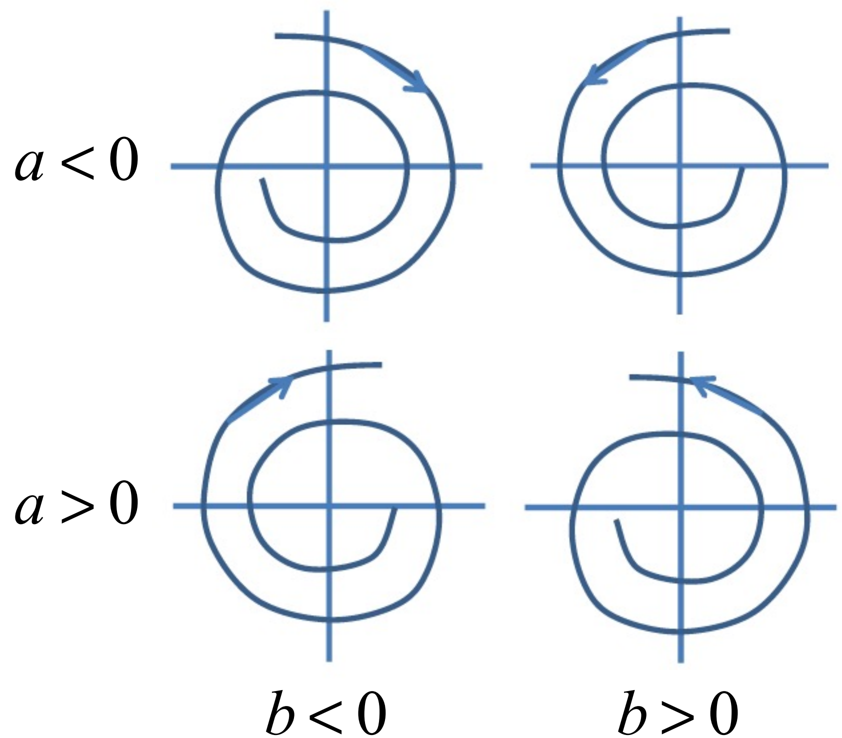
- If the eigenvalues are generally complex, then the normal form is

$$\mathbf{V}'\mathbf{A}(\mathbf{V}')^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

- Recall the eigenvalues are $\lambda_1 = a + jb$; $\lambda_2 = a - jb$

- Phase portraits are **spirals**

$a < 0$: spiral is stable
$a > 0$: spiral is unstable
$b > 0$: spiral is anticlockwise
$b < 0$: spiral is clockwise



Phase space equation summary

- We have drawn diagrams showing how solutions behave in normal forms of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$:

$$\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{y} \quad \Rightarrow \quad \mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \mathbf{y}_0$$

real; complete eigenvector set

$$\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \mathbf{y} \quad \Rightarrow \quad \mathbf{y}(t) = e^{\lambda_1 t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{y}_0$$

degenerate

$$\dot{\mathbf{y}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{y} \quad \Rightarrow \quad \mathbf{y}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{y}_0$$

complex

Example of coordinate change

- Consider the system $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{x}$

- Eigenvalues from characteristic eq: $\lambda^2 + \lambda - 2 = 0 \;\Rightarrow\; \lambda \in \{1, -2\}$

- Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

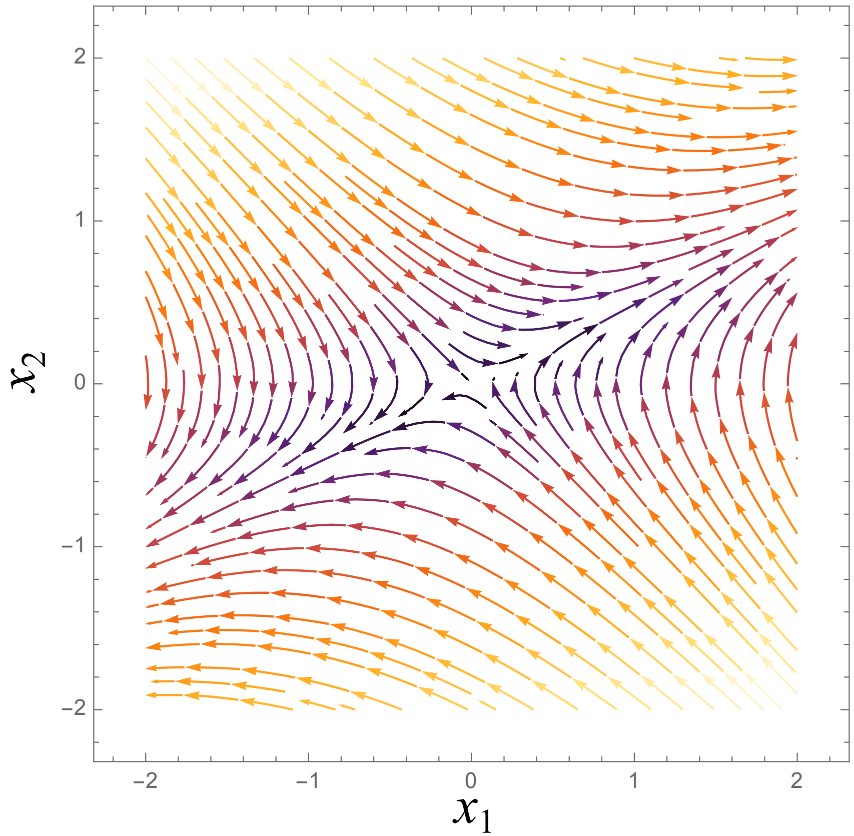
- Define new coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$, which place transformed axes in $[2\ \ 1]^{\mathrm{T}}$ (unstable) direction and $[-1\ \ 1]^{\mathrm{T}}$ (stable) direction

Example continued

- Continue with $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}$

- Solution is $\mathbf{x}(t) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}_0$

$$\mathbf{x}(t) = \frac{1}{3} \begin{bmatrix} e^{-2t} + 2e^{t} & -2e^{-2t} + 2e^{t} \\ -e^{-2t} + e^{t} & 2e^{-2t} + e^{t} \end{bmatrix} \mathbf{x}_0$$



Linearization of nonlinear systems

- Consider the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with equilibrium at $\mathbf{x}^*$
- Let $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$ and assume that $\mathbf{f}$ is differentiable.
  Then the Taylor expansion of the ith entry, f_i, of $\mathbf{f}$ gives

$$f_i(\mathbf{x}^* + \mathbf{w}) = f_i(\mathbf{x}^*) + \sum_{j=1}^{n} \left(\left. \frac{\partial f_i}{\partial x_j} \right|_{\mathbf{x}^*} w_j + O(|w_j|^2) \right)$$

- Noting that equilibrium is independent of time by definition

$$\dot{\mathbf{x}} = \dot{\mathbf{w}} = D\mathbf{f}(\mathbf{x}^*)\mathbf{w} + O(\|\mathbf{w}\|^2)$$

- The matrix $D\mathbf{f}(\mathbf{x})$ is called the **Jacobian** of the vector-valued function $\mathbf{f}$, with ijth entry $(D\mathbf{f})_{ij} = \partial f_i / \partial x_j$

Hyperbolic equilibria

Definition (hyperbolic equilibrium): If $\mathbf{x}^*$ is an equilibrium of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then $\mathbf{x}^*$ is called a *hyperbolic fixed point* if all eigenvalues of the Jacobian $D\mathbf{f}(\mathbf{x}^*)$ have nonzero real parts

- This leads to an important **theorem**: if an equilibrium point is a hyperbolic fixed point and all the eigenvalues of the Jacobian have negative real parts, then the equilibrium solution $\mathbf{x} = \mathbf{x}^*$ is asymptotically stable

- Note that this proves asymptotic stability but does not say anything about the size of the region of stability
  – this depends on the size of δ from the game on slide 7

- In lecture 3 we will see how linearizations near equilibrium points can be used to get information about nonlinear systems

Example: Duffing oscillator

The Duffing oscillator is described for $\gamma \geq 0$ by

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - \gamma y$$

- equilibria: $(x^*, y^*) = (0,0)$ and $(\pm 1, 0)$
- Jacobian: $D\mathbf{f} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -\gamma \end{bmatrix}$
- at $(0,0)$: $\quad \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4}}{2} \quad \Longrightarrow \quad$ unstable

  at $(\pm 1, 0)$: $\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 8}}{2} \quad \Longrightarrow \quad$ asymptotically stable if $\gamma > 0$
- If $\gamma = 0$, $(\pm 1, 0)$ is a centre $\Longrightarrow$ local linearization inconclusive

Example: Duffing oscillator

Phase plane of the Duffing oscillator with $\gamma = 0.5$