

The background features a large, faint watermark of the University of Oxford seal. The seal is circular and contains the text 'UNIVERSITY OF OXFORD' around the perimeter. In the center, there is a shield with an open book, and the Latin motto 'DOMINA NVS IO' is visible on the book's pages. Above the shield are two crowns, and below it is a single crown. The seal is rendered in a light beige color.

## Lecture 3: Invariant manifolds

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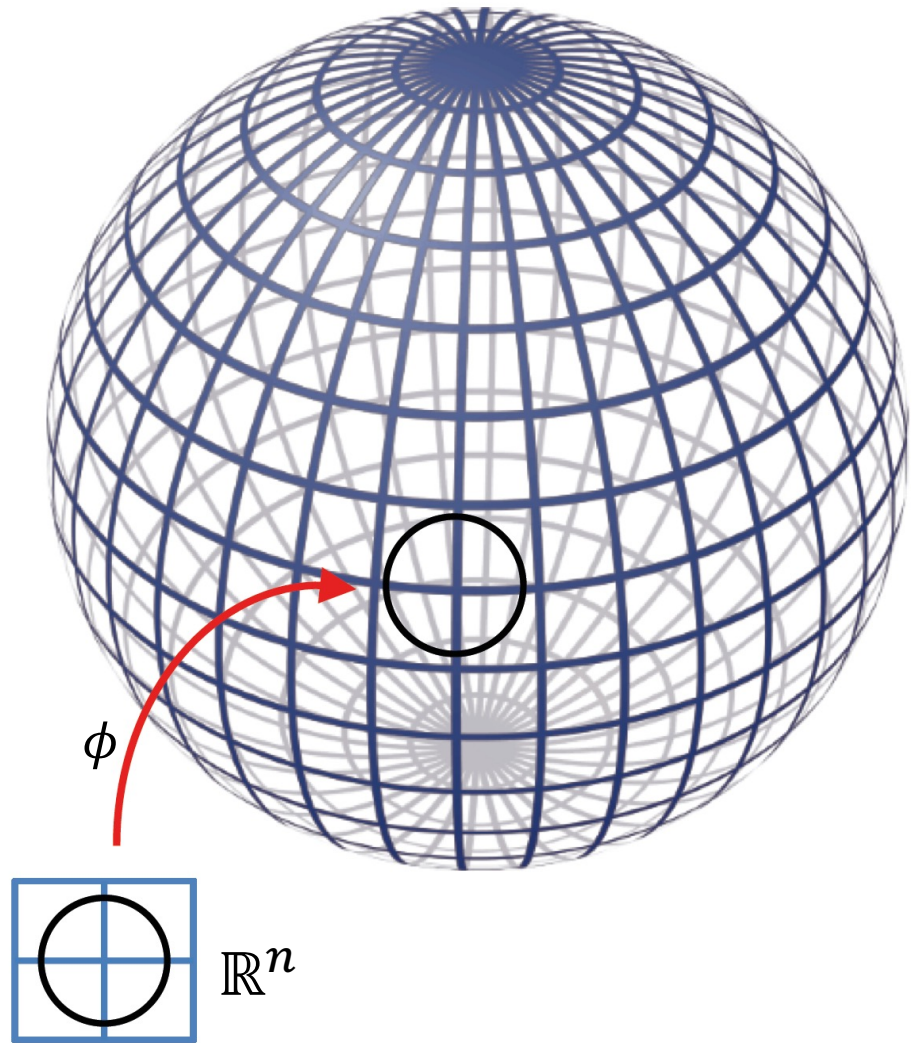
# Lecture 3 overview

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- This lecture focuses on the topic of **differential manifolds**,  
– smooth surfaces that locally behave like a Euclidean space
- We are interested in paths on these manifolds and how they characterize the solutions of differential equations; that is, we are interested in differential manifolds in **phase space**
- We will again touch on hyperbolic equilibria, and also consider the manifolds associated with **nonhyperbolic equilibria**
- The first important idea is a **vector subspace** – a concept that generalizes the ideas of, say, lines within planes; manifolds extend this idea by allowing space to be curved

# Differential manifolds

- A manifold  $M$  is locally made up of patches copied from  $\mathbb{R}^n$
- e.g. here  $M$  is the surface of a sphere and the patches are planes
- The definition of a differential manifold involves how different coordinate systems are placed on  $M$ , and how neighboring patches transform into each other



# Linear systems: stable, unstable, & centre subspaces

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- Last lecture we studied the  $n$ -dimensional linear system

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}\mathbf{w}$$

- Coefficient matrix  $\mathbf{A}$  generally has three kinds of eigenvalues:

- a stable set, such that  $\text{Re}\{\lambda\} < 0$ :  $\lambda_1^S, \lambda_2^S, \dots, \lambda_s^S$

- an unstable set, such that  $\text{Re}\{\lambda\} > 0$ :  $\lambda_1^U, \lambda_2^U, \dots, \lambda_u^U$

- a centre set, such that  $\text{Re}\{\lambda\} = 0$ :  $\lambda_1^C, \lambda_2^C, \dots, \lambda_c^C$

- Since there are  $n$  eigenvalues in total,  $s + u + c = n$

# The eigenvector basis

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- If all the eigenvalues of  $\mathbf{A}$  are real and distinct, its  $n$  eigenvectors span phase space, so any vector in the space can be expressed as a weighted sum of the eigenvectors  $\mathbf{v}_i$

- In particular, given initial condition  $\mathbf{w}(0)$ , we can write

$$\mathbf{w}(0) = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$

and the linear system will be solved by

$$\mathbf{w}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$$

- If the eigenvalues are complex or  $\mathbf{A}$  is degenerate, we have other ways of defining  $\mathbf{v}_i$ , for example through normal forms

# Bases for the subspaces

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- For linear systems phase space can generally be split into **three subspaces** spanned by different sets of eigenvectors:

$\{\mathbf{v}_i^S\}_{i=1}^s$ : associated eigenvalues have negative real parts (stable)

$\{\mathbf{v}_j^U\}_{j=1}^u$ : associated eigenvalues have positive real parts (unstable)

$\{\mathbf{v}_k^C\}_{k=1}^c$ : associated eigenvalues have null real parts (centre)

- Each vector, multiplied by the appropriate function of time determined by the normal form, creates a component of the solution, summarized as  $\{\mathbf{w}_i^S(t), \mathbf{w}_j^U(t), \mathbf{w}_k^C(t)\}_{i,j,k}$

$$\mathbf{w}(t) = \sum_{i=1}^s c_i^S \mathbf{w}_i^S(t) + \sum_{j=1}^u c_j^U \mathbf{w}_j^U(t) + \sum_{k=1}^c c_k^C \mathbf{w}_k^C(t)$$

- These three components are written  $\mathbf{w}(t) = \mathbf{w}^S(t) + \mathbf{w}^U(t) + \mathbf{w}^C(t)$

# Effect of a change of coordinates

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- Last lecture we considered changes of coordinates  $\mathbf{y} = \mathbf{K}^{-1}\mathbf{x}$
- Let the transformation  $\mathbf{T}$  be made up columnwise of the sets of constant eigenvectors we just identified:  $\{\mathbf{v}_i^S, \mathbf{v}_j^U, \mathbf{v}_k^C\}_{i,j,k}$
- Let  $\mathbf{z} = \mathbf{T}^{-1}\mathbf{w}$  be the new coordinates:  $\mathbf{z} = \mathbf{T}^{-1}\mathbf{w} = \begin{bmatrix} \mathbf{z}_S \\ \mathbf{z}_U \\ \mathbf{z}_C \end{bmatrix}$
- $\mathbf{T}$  transforms the coefficient matrix  $\mathbf{A}$  into a block diagonal matrix whose diagonal entries are normal forms:

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{A}_S & 0 & 0 \\ 0 & \mathbf{A}_U & 0 \\ 0 & 0 & \mathbf{A}_C \end{bmatrix}$$

# New equations of motion

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- Change coordinates of the governing system:

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}\mathbf{w} \Rightarrow \mathbf{T} \frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{T}\mathbf{z} \Rightarrow \frac{d\mathbf{z}}{dt} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} \Rightarrow \frac{d\mathbf{z}}{dt} = \begin{bmatrix} \mathbf{A}_S & 0 & 0 \\ 0 & \mathbf{A}_U & 0 \\ 0 & 0 & \mathbf{A}_C \end{bmatrix} \mathbf{z}$$

- Since the transformed coordinates separate the eigenvectors associated with each subspace, this breaks the equations of motion into three decoupled systems:

$$\frac{d}{dt} \begin{bmatrix} \mathbf{z}_S \\ \mathbf{z}_U \\ \mathbf{z}_C \end{bmatrix} = \begin{bmatrix} \mathbf{A}_S & 0 & 0 \\ 0 & \mathbf{A}_U & 0 \\ 0 & 0 & \mathbf{A}_C \end{bmatrix} \begin{bmatrix} \mathbf{z}_S \\ \mathbf{z}_U \\ \mathbf{z}_C \end{bmatrix} \Rightarrow \begin{aligned} \frac{d\mathbf{z}_S}{dt} &= \mathbf{A}_S\mathbf{z}_S \\ \frac{d\mathbf{z}_U}{dt} &= \mathbf{A}_U\mathbf{z}_U \\ \frac{d\mathbf{z}_C}{dt} &= \mathbf{A}_C\mathbf{z}_C \end{aligned}$$



# Consequences of the coordinate change

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- By introducing normal forms and changing coordinates we have split the solutions of the original governing system into three distinct **subspaces**:  $E^S$ ,  $E^U$ , and  $E^C$
- If a solution starts in one of these subspaces, its trajectory will stay there; it will not cross into one of the other subspaces
- Since they are separated in this way, the subspaces of the  $n$ -dimensional first-order linear autonomous problem

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}\mathbf{w}$$

are said to be **invariant** with respect to the flow  $e^{\mathbf{A}t}$

# Example: decomposing into subspaces

Consider  $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}$

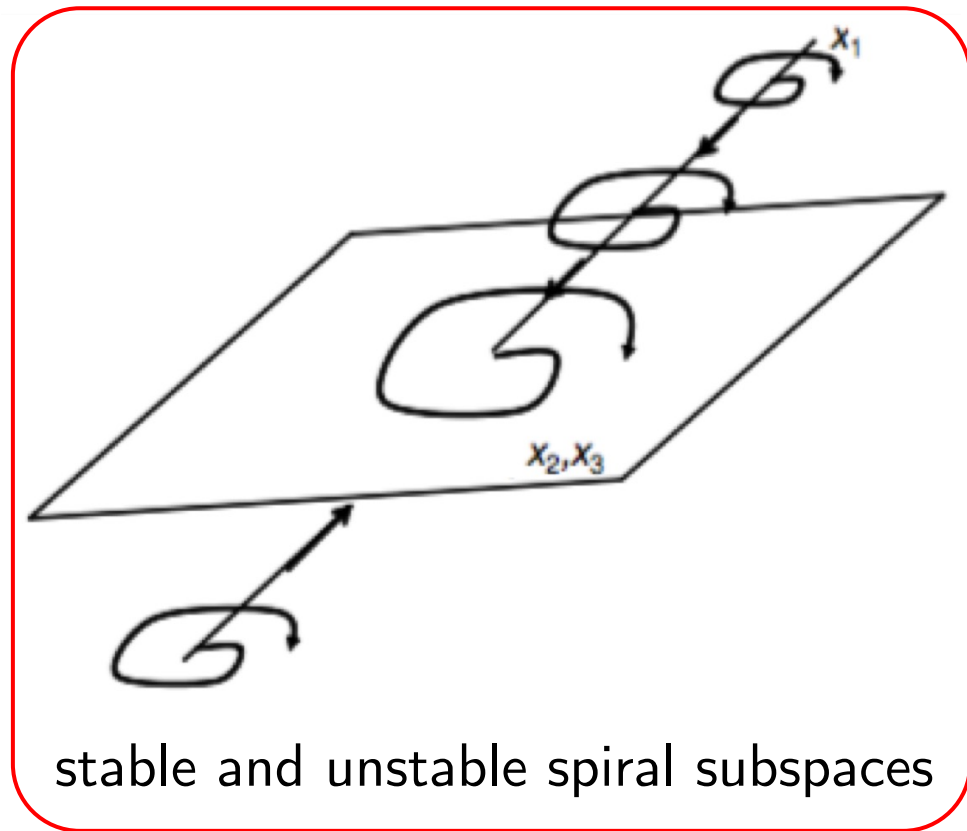
- Eigenvalues:  $\lambda \in \{-3, 2 \pm j\}$

- Eigenvectors:

$$\mathbf{v} \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1+j \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1-j \\ 1 \end{bmatrix} \right\}$$

- Solution:

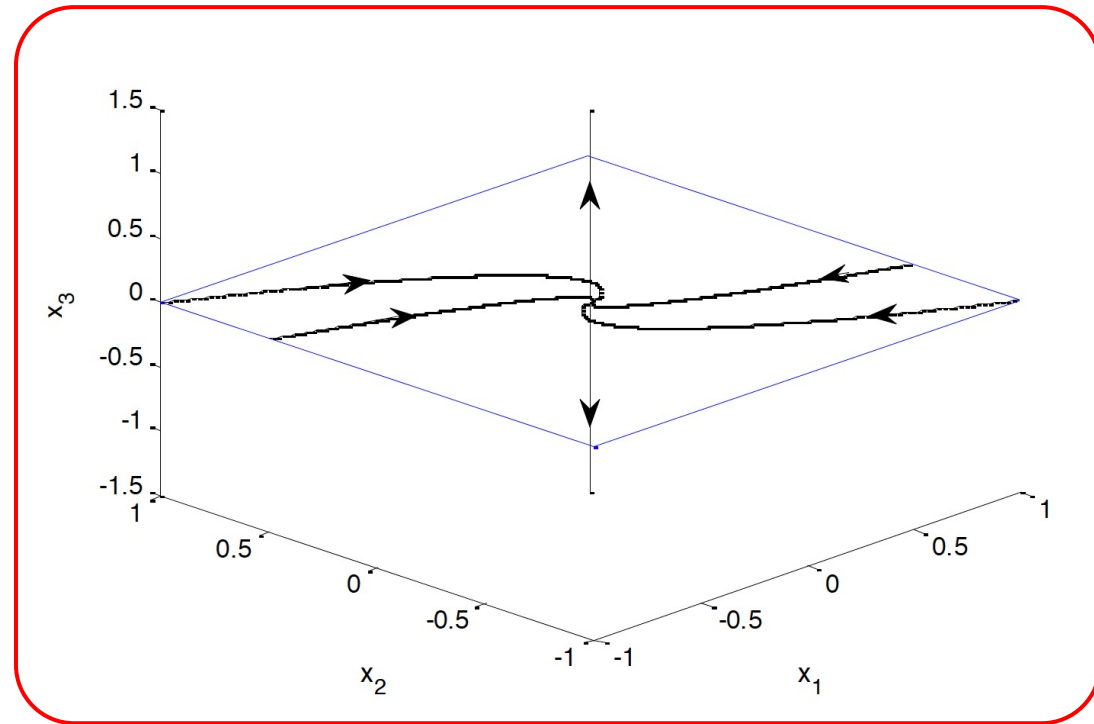
$$\mathbf{x}(t) = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}(\cos t + \sin t) & -2e^{2t} \sin t \\ 0 & e^{2t} \sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} \mathbf{x}(0)$$



# Example: a degenerate subspace

Consider  $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \gamma \end{bmatrix} \mathbf{x}$  with  $\lambda < 0$  and  $\gamma > 0$

- There is a repeated eigenvalue ( $\lambda$ )
- The top left 2x2 block is degenerate
- Here  $\mathbf{x}_3$  is an unstable subspace and  $\{\mathbf{x}_1, \mathbf{x}_2\}$  span a stable subspace



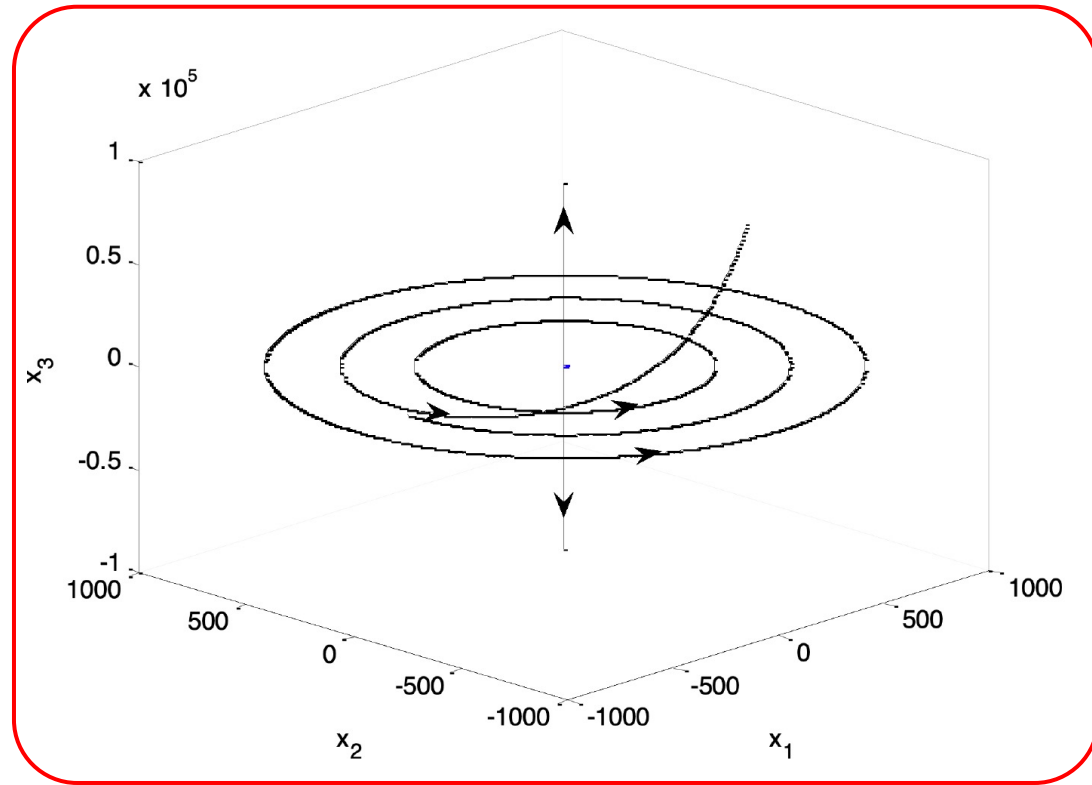
# Example: a centre subspace

Consider  $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$

- Eigenvalues:  $\lambda \in \{\pm j, 2\}$

- Eigenvectors:

$$\mathbf{v} \in \left\{ \begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$



- Here  $\mathbf{x}_3$  is an unstable subspace;  
and  $\{\mathbf{x}_1, \mathbf{x}_2\}$  plane is a centre subspace

# Observations from the examples

- All points in the **stable subspace** flow along streamlines that end up at the origin

$$\lim_{t \rightarrow \infty} \mathbf{w}^S = 0$$

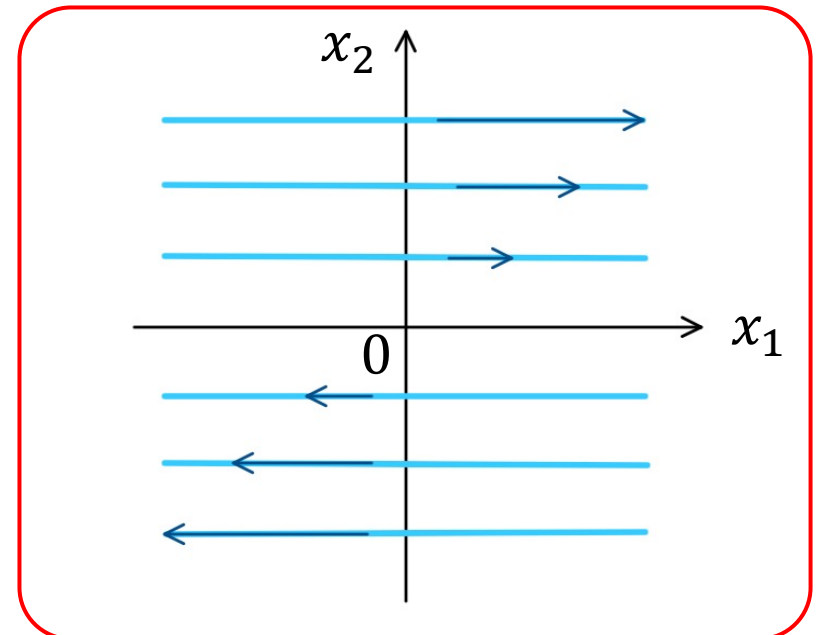
- All points in the **unstable subspace** flow away from the origin; if you go 'upstream' along the flow you end up at the origin

$$\lim_{t \rightarrow -\infty} \mathbf{w}^U = 0$$

- No simple generalizations for **centre subspaces**

– consider for example

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}$$



# Hartman–Grobman theorem: introduction

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- Now that we have trained our intuition with linear problems, we can extend our perspective to the nonlinear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

- We can gain insight about stability by considering the **linearization** of this system near an equilibrium point  $\mathbf{x}^*$
- The **Hartman–Grobman** theorem justifies the process of drawing qualitative conclusions about how dynamical systems behave near to hyperbolic equilibria by examining local linearizations
- Note: nothing is said about non-hyperbolic equilibria

# Local theory for nonlinear systems

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- Given the nonlinear autonomous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

- Linearize about an equilibrium point  $\mathbf{x}^*$  by setting  $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$ :

$$\frac{d\mathbf{w}}{dt} = D\mathbf{f}(\mathbf{x}^*)\mathbf{w} + O(\|\mathbf{w}\|^2)$$

- The linearized system phase space is split into 3 subspaces by a coordinate transformation based on eigenvectors of  $D\mathbf{f}(\mathbf{x}^*)$ :

$$\frac{d\mathbf{z}_S}{dt} = \mathbf{A}_S\mathbf{z}_S + \mathbf{R}_S(\mathbf{z})$$

$$\frac{d\mathbf{z}_U}{dt} = \mathbf{A}_U\mathbf{z}_U + \mathbf{R}_U(\mathbf{z})$$

$$\frac{d\mathbf{z}_C}{dt} = \mathbf{A}_C\mathbf{z}_C + \mathbf{R}_C(\mathbf{z})$$

- Each  $\mathbf{A}$  is a square real matrix (stable, unstable, or centre)

- The  $\mathbf{R}$ 's are vector functions that quantify the error

# The Hartman–Grobman theorem

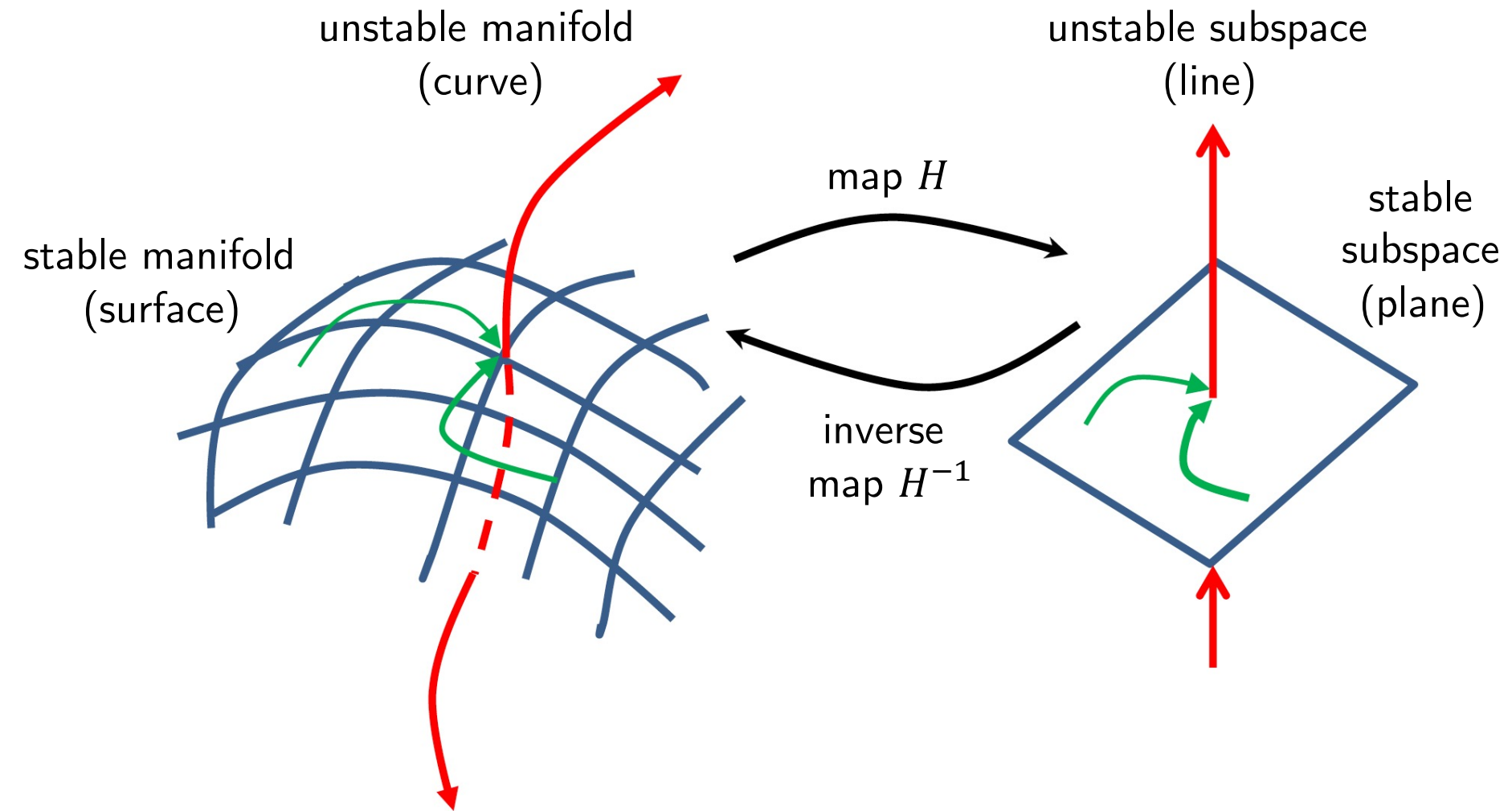
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**Theorem:** For each **hyperbolic equilibrium** point, there exists a bi-continuous function  $H$  (a mapping that is continuous and whose inverse is also continuous) between an open set containing the equilibrium point and an open set containing the origin of the linearized model, such that the trajectories are mapped exactly and the parameterization of time is preserved

- Near the origin, the stable linear subspace is mapped to a stable manifold in the region surrounding the equilibrium point
- Near the origin, the unstable linear subspace is mapped to an unstable manifold in the region around the equilibrium point
- Nothing is said about centre subspaces



# Illustration

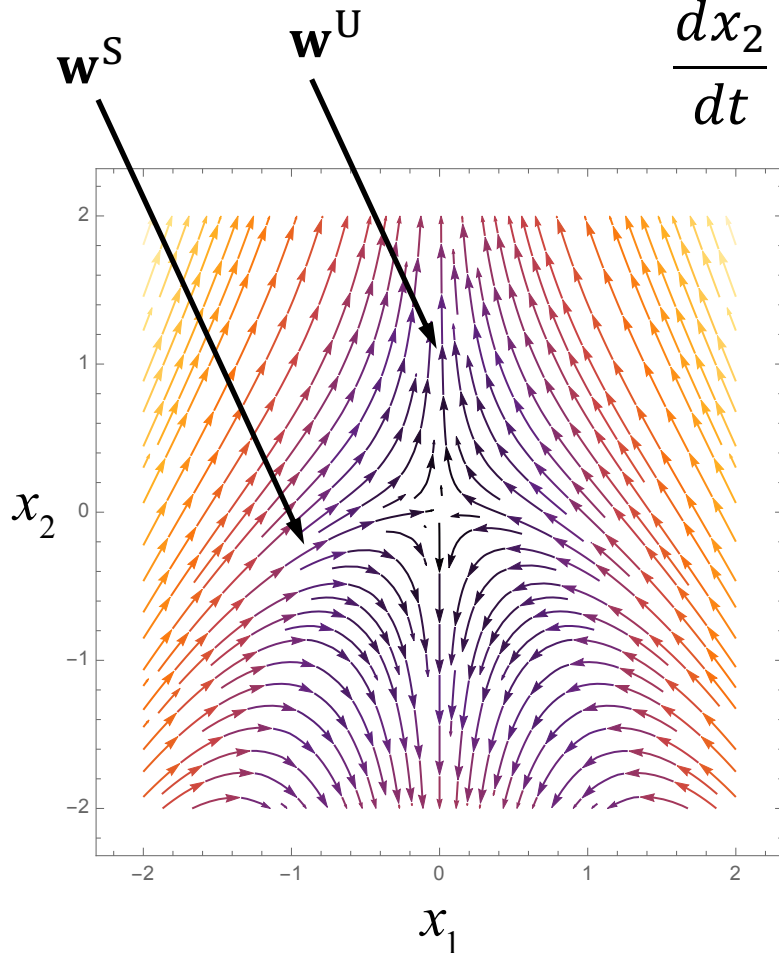


# Example of Hartman–Grobman

Consider the autonomous nonlinear system:

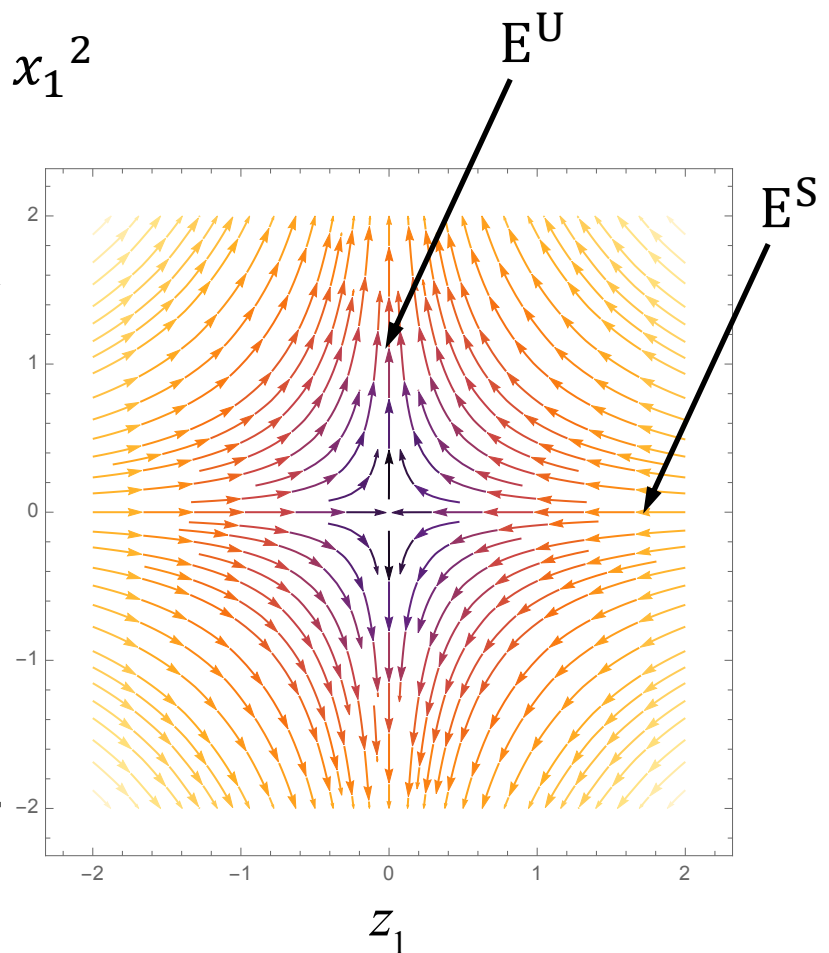
$$\frac{dx_1}{dt} = -x_1$$

$$\frac{dx_2}{dt} = x_2 + x_1^2$$



$H$

$H^{-1}$



# Non-hyperbolic equilibria

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- We cannot say very much about non-linear equilibria whose linearized models are centres
- There are some tricks available...  
e.g. a 2D system can be transformed into **polar coordinates**:

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

Let  $x_1 = r \cos \theta$  and  $x_2 = r \sin \theta$ , then

$$\dot{r} = \frac{x_1 \dot{x}_1 + x_2 \dot{x}_2}{r}, \quad \dot{\theta} = \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{r^2}$$

- We can now ask: does the radius grow, shrink, or stay constant?

# Polar transformation example

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Consider the system

$$\dot{x} = -y - xy$$

$$\dot{y} = x + x^2$$

- A move into polar coordinates shows that

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{-x(y + xy) + y(x + x^2)}{r} = 0$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x(x + x^2) + y(y + xy)}{r^2} = \frac{(1 + x)(x^2 + y^2)}{r^2} = 1 + x$$

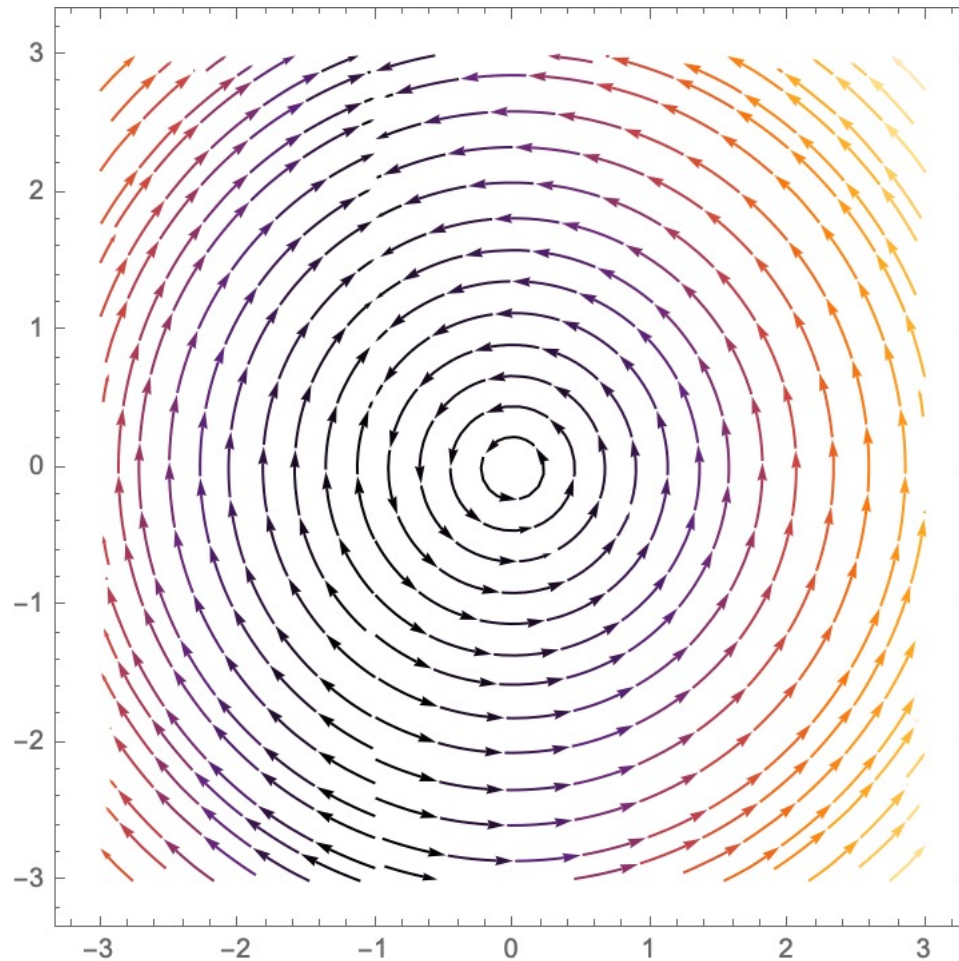
- This has a **nonlinear centre** at the origin since  $r = \text{constant}$

# Polar transformation example

Polar coordinates:

$$\dot{r} = 0$$
$$\dot{\theta} = 1 + x$$

Phase portrait:



# Symmetric systems

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- Another idea that can be used to understand non-hyperbolic equilibrium points is **symmetry**
- A 2D nonlinear system with state  $(x, y)$  is symmetric with respect to the  $x$ -axis if it is invariant under the transformation

$$(t, y) \rightarrow (-t, -y)$$

- **If:** the system is symmetric with respect to either  $x$  or  $y$ ,  
**and:** the origin is an equilibrium point  
**then:** centres map to centres between the nonlinear system and its linear approximation

# Symmetry example

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Consider the system

$$\dot{x} = y - y^3 = f(x, y)$$

$$\dot{y} = -x - y^2 = g(x, y)$$

- The equilibrium point at the origin has Jacobian  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and is therefore a linear centre

- Functions  $f$  and  $g$  satisfy:
$$\begin{aligned} f(x, -y) &= -f(x, y) \\ g(x, -y) &= g(x, y) \end{aligned}$$

- But 
$$\frac{dx}{d(-t)} = f(x, -y) \Rightarrow (x(t), y(t)) = (x(-t), -y(-t))$$
$$\frac{d(-y)}{d(-t)} = g(x, -y)$$

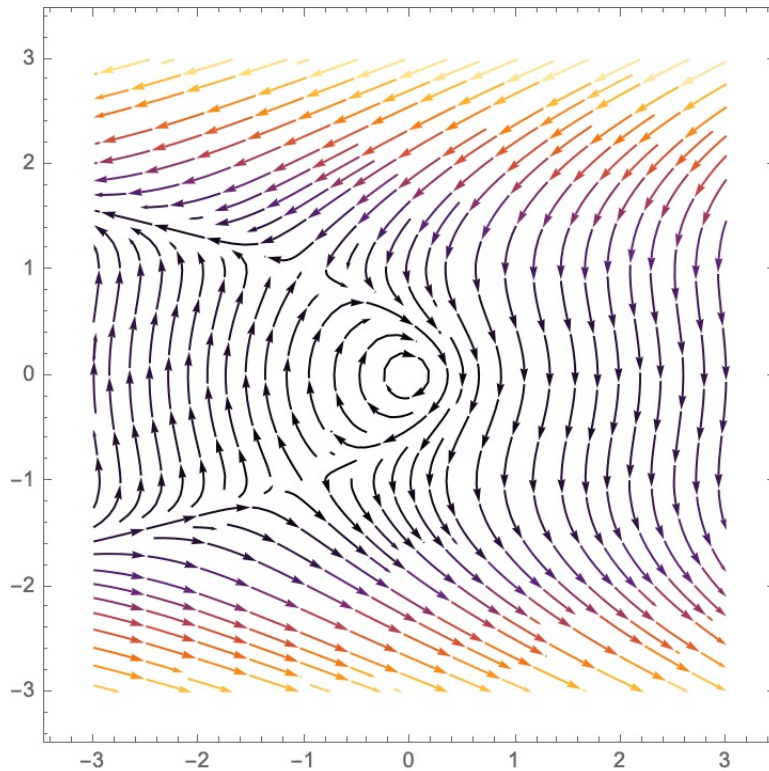
so the nonlinear system also has a centre at the origin

# Symmetry example

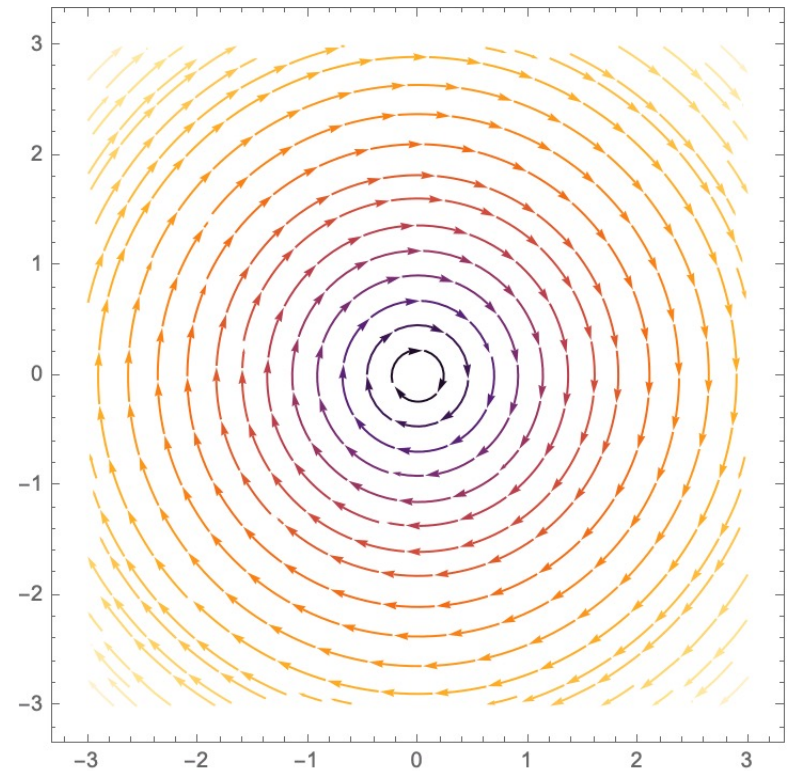
$$\dot{x} = y - y^3 = f(x, y)$$

$$\dot{y} = -x - y^2 = g(x, y)$$

$$\Rightarrow (x(t), y(t)) = (x(-t), -y(-t))$$



Nonlinear centre at (0,0)



Linearised system



# Conservative systems

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- If there exists a nonconstant function  $V(\mathbf{x})$  such that  $dV/dt = 0$  along solutions of the nonlinear differential equation  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ , the equations are called **conservative**
- The **potential function**  $V(\mathbf{x})$  has the property that it does not change along the solution trajectories
- If  $\mathbf{x} = \mathbf{x}^*$  is an isolated equilibrium point and there is a potential function  $V(\mathbf{x})$  that has a local minimum or maximum at  $\mathbf{x}^*$ , then there is a region around that point that contains a **closed orbit**

# Conservative example

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Consider the system

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= f(x)\end{aligned}$$

- Multiply the second equation by the first to form  $-f(x)\dot{x} + v\dot{v} = 0$

- Integrate to get  $-\int_{x_0}^x f(s)ds + \frac{v^2}{2} = \text{constant}$

- Thus potential energy + kinetic energy is constant

- Such systems are called **Newtonian** dynamical systems

# A more general approach

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- Any system of the form

$$\frac{dx}{dt} = f(x)g_1(y)$$

$$\frac{dy}{dt} = f(y)g_2(x)$$

- We can rearrange to

$$\frac{g_2(x) dx}{f(x) dt} - \frac{g_1(y) dy}{f(y) dt} = 0 = \frac{dV}{dt}$$

- This can be integrated to obtain the potential function  $V(x, y)$

# Final conservative example

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Consider the system  $\dot{x} = x - xy = x(1 - y)$   
 $\dot{y} = -y + xy = y(x - 1)$

- There are equilibrium points at  $(0,0)$  and  $(1,1)$
- Jacobian at  $(0,0)$ :  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  Hartman–Grobman says this is a **nonlinear saddle point**
- Jacobian at  $(1,1)$ :  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  is this a nonlinear centre?
- $\frac{x-1}{x}\dot{x} - \frac{1-y}{y}\dot{y} = 0 \Rightarrow x - \ln x + y - \ln y = V(x, y) = \text{constant}$
- At  $(1,1)$ :  $\partial V/\partial x = \partial V/\partial y = 0$  and  $\partial^2 V/\partial x \partial y$  has eigenvalues  $> 0$   
 $\Rightarrow$  this is a minimum point of  $V(x, y)$   
 $\Rightarrow (1,1)$  is a **nonlinear centre**

# Final conservative example

$$\begin{aligned}\dot{x} &= x - xy = x(1 - y) \\ \dot{y} &= -y + xy = y(x - 1)\end{aligned}$$

