Lecture 5: Asymptotic behaviour

Mark Cannon
mark.cannon@eng.ox.ac.uk

Lecture 5 overview

- We now consider how trajectories behave at the 'beginning' and 'end' of time – their asymptotic behaviour

- To formalize this discussion we will introduce key concepts relating to flow trajectories:
  limit points, **limit sets**, and various types of **orbit**

- We will also characterize domains in phase space through the concepts of **invariance**, **attractors**, and **basins of attraction**

- Finally we will characterize various types of attractors with the **Poincaré-Bendixson theorem**, which fully characterizes positively invariant regions of the phase plane

Limit points

Global properties of trajectories:

- Suppose $\phi(t, \mathbf{x}_0)$ is the flow of $\mathbf{f}(\mathbf{x})$ with $\phi(0, \mathbf{x}_0) = \mathbf{x}_0$
  (i.e. $\mathbf{x}(t) = \phi(t, \mathbf{x}_0)$ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\phi(0, \mathbf{x}_0) = \mathbf{x}_0$)

- This solution defines a path or trajectory in some set $D \subseteq \mathbb{R}^n$ containing $\mathbf{x}_0$

$$\Gamma = \{\mathbf{x} \in D : \mathbf{x} = \phi(t, \mathbf{x}_0), t \in \mathbb{R}\}$$

- We want to determine the asymptotic behaviour of this solution – the α and ω **limit points** of the trajectory

ω limit points

Definition: A point $\mathbf{p} \in D$ is called an ω **limit point** of the trajectory $\phi(t, \mathbf{x})$ if there exists a sequence of times $\{t_0, t_1, \ldots\}$, with $t_i \to \infty$ as $i \to \infty$, such that

$$\lim_{i \to \infty} \phi(t_i, \mathbf{x}) = \mathbf{p}$$

this point is denoted $\omega(\mathbf{x})$

Note that we may need to choose the times $\{t_i,\ i = 0, 1, \ldots\}$ carefully to define a limit point

e.g. for any $x \in \mathbb{R}$, $\phi(t, x) = e^{-2t}x + \sin(t)$ has the limit points

$$\omega(x) = \begin{cases} 0 & \text{if } t_i = i\pi \\ 1 & \text{if } t_i = (i + \frac{1}{2})\pi \end{cases}$$

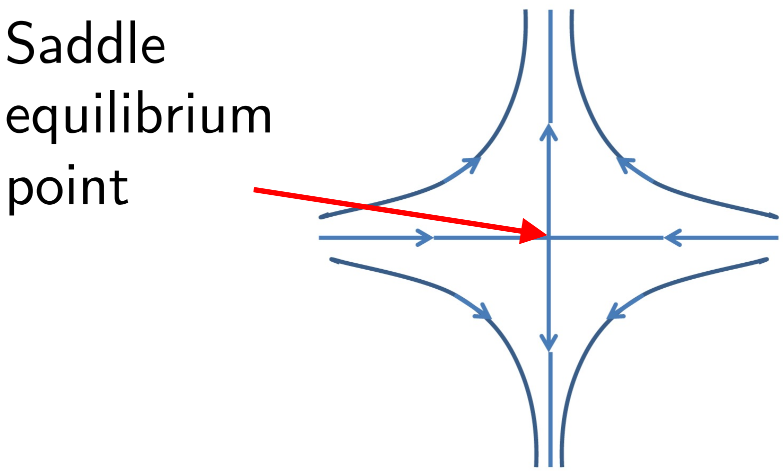
α limit points

Definition: A point $\mathbf{p} \in D$ is called an α **limit point** of the trajectory $\phi(t, \mathbf{x})$ if there exists a sequence of times $\{t_0, t_1, \ldots\}$, with $t_i \to -\infty$ as $i \to \infty$, such that

$$\lim_{i \to \infty} \phi(t_i, \mathbf{x}) = \mathbf{p}$$

this point is denoted $\alpha(\mathbf{x})$

Example: a saddle point equilibrium is an ω limit for any point on the stable manifold, and an α limit for any point on the unstable manifold



Limit sets

A trajectory Γ may have multiple α and/or ω limit points

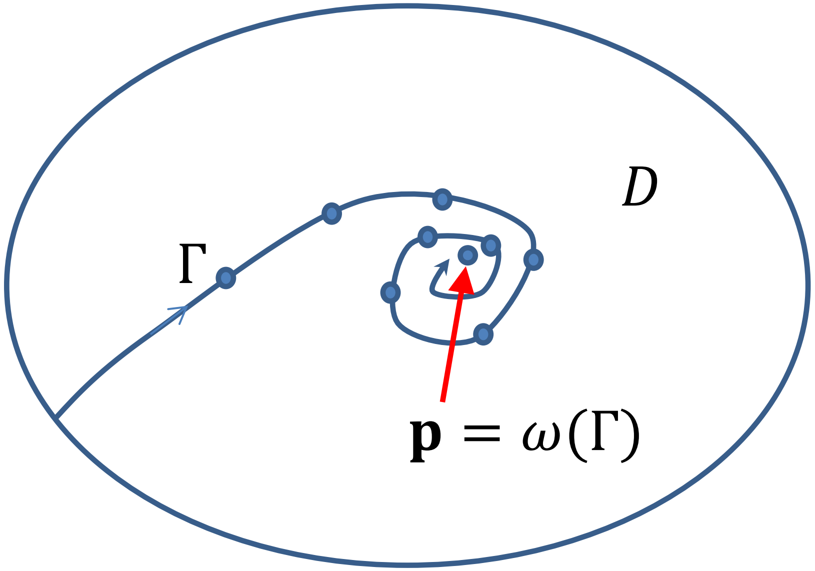
Definition: The α **limit set** and ω **limit set**, denoted $\alpha(\Gamma)$ and $\omega(\Gamma)$, are the sets of all α limit points and ω limit points for the trajectory Γ

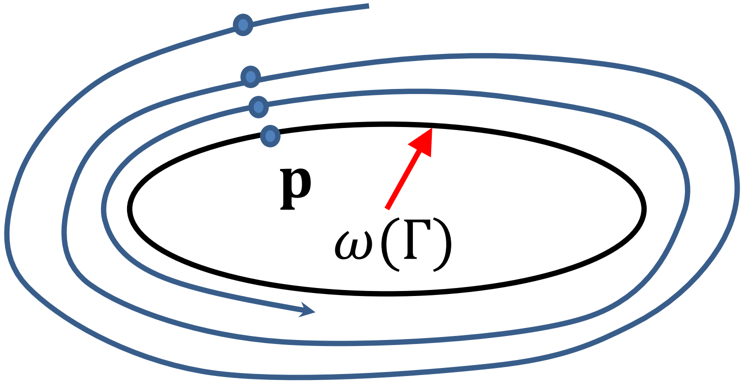
Note that:

- $\alpha(\Gamma)$ is the set of points from which the trajectory Γ originates (in the limit as $t \to -\infty$) and $\omega(\Gamma)$ is the set of points to which it tends (in the limit as $t \to \infty$)

- The set of all limit points is called the **limit set** of Γ

Examples





A sequence of points leading to an ω limit set $\omega(\Gamma)$ consisting of an isolated point $\mathbf{p}$

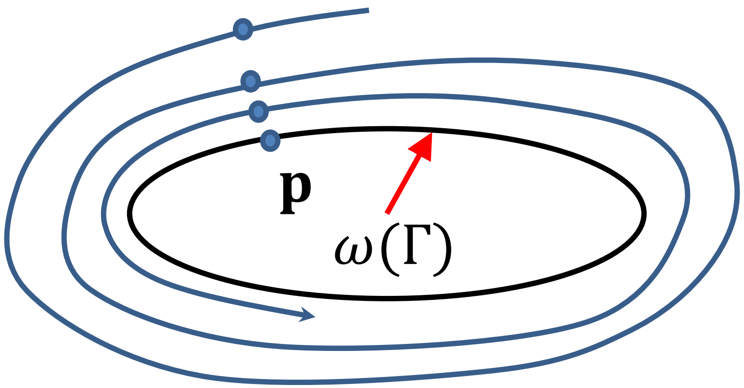
A sequence of points converging to an ω limit point $\mathbf{p}$ when the limit set $\omega(\Gamma)$ is not an isolated point

Equilibrium points

- An equilibrium point $\mathbf{x}^*$ is its own α and ω limit point

  Conversely if a trajectory has a unique ω limit point $\mathbf{x}^*$, then $\mathbf{x}^*$ is an equilibrium point

- Not all ω limit points are equilibrium points – e.g. an ω limit point $\mathbf{p}$ on an orbiting trajectory $\omega(\Gamma)$



- If a point $\mathbf{p}$ is a limit point and $\dot{\mathbf{p}} \neq 0$, the trajectory is a closed orbit

  For closed orbits:
  – the sequence of points must be picked carefully
  – there are infinitely many points in the ω limit set

Invariance

Definition: Let $\phi(t, \mathbf{x})$ be the flow of $\mathbf{f}(\mathbf{x})$ on a domain D; then a set $S \subset D$ is called **positively invariant** if

$$\phi(t, \mathbf{x}) \in S \quad \text{for all } \mathbf{x} \in S \text{ and all } t \geq 0$$

- All points in S stay in S under the action of the flow – the solution cannot 'escape' from S

  (we saw an example of this in the case of the stable and unstable invariant manifolds in earlier lectures)

- If a region M is positively invariant, closed and bounded, then the ω limit set is non-empty (all flows have to go somewhere!)

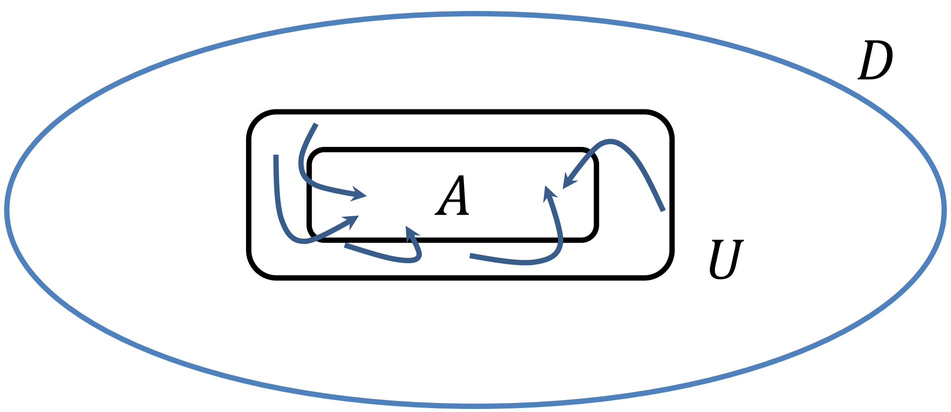
- A limit set is necessarily positively invariant

Attraction

- A **neighbourhood** is a set of points surrounding a given point $\mathbf{x}$ such that the distance from $\mathbf{x}$ to any point in the set is less than some positive number ε

- An invariant set $A \subset D$ is called **attracting** if:
  1. There is some neighbourhood U of A that is positively invariant
  2. All trajectories starting in U converge to A as $t \to \infty$

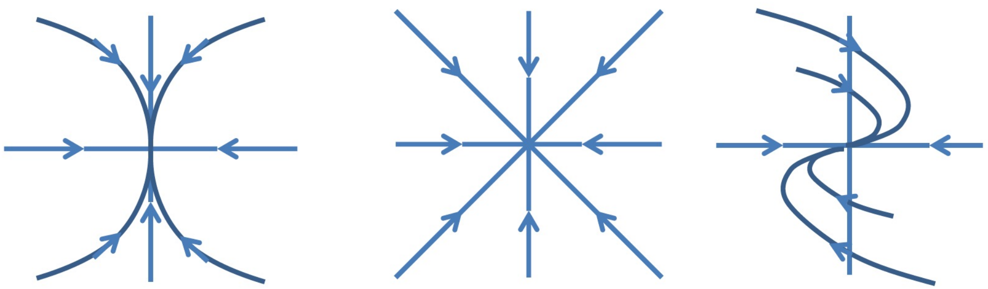
- The neighbourhood U so defined is a **trapping region** of A



Attractor

- An **attractor** is an invariant attracting set (e.g. a limit cycle or equilibrium point) such that no subset of the invariant set is itself an invariant attracting set

- A stable node or focus is an attractor, because it is the ω-limit set of all trajectories that pass through points in a neighborhood of the equilibrium point



- A saddle point is not an attractor, because trajectories with unstable components leave the saddle point's neighborhood

Example

Consider the dynamical system

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$

Transform to polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

- $r = 0$ is an unstable hyperbolic equilibrium point

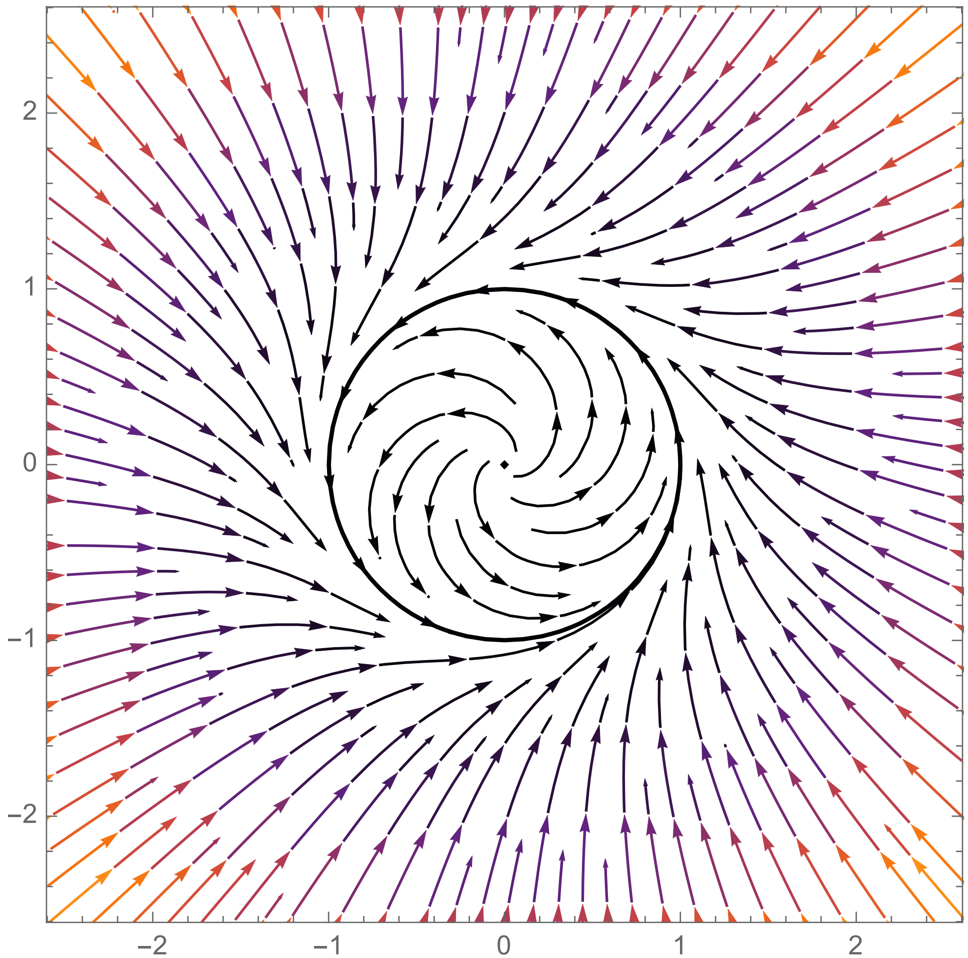
  $r = 1$ is a limit cycle since $\dot{r} = 0$

- to evaluate the stability of the limit cycle, set $r = 1 + \delta r$:

$$\dot{r} = r(1 - r^2) \;\Rightarrow\; \frac{d}{dt}(1 + \delta r) = (1 + \delta r)[1 - (1 + \delta r)^2]$$
$$\Rightarrow\; \frac{d}{dt}(\delta r) = -2\delta r + O(\delta r^2)$$

  so the limit cycle is stable

Solution trajectories:



$r(1-r^2) < 0 \quad \text{if } r > 1$

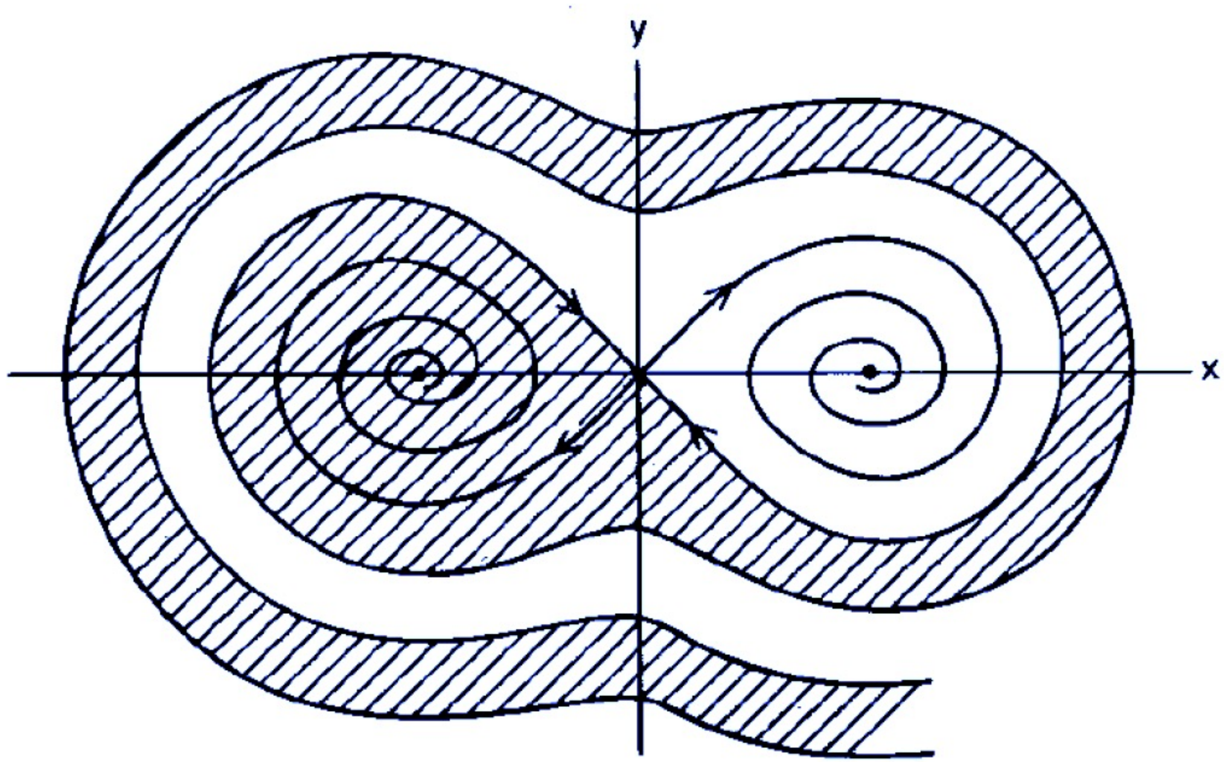
$r(1-r^2) = 0 \quad \text{if } r \in \{0.1\}$

$r(1-r^2) > 0 \quad \text{if } 0 < r < 1$

- $r = 1$ is the ω limit set for all points in the plane except the origin (which is its own ω limit set)
- the trapping region is the whole plane, excluding the origin

Basin of attraction

The **domain** (or **basin**) **of attraction** of an attracting set A is the union of all trajectories that form a trapping region of A



The domain of attraction for the leftmost equilibrium point $(-1,0)$ of the Duffing oscillator

LaSalle's invariance principle

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ and let V be a continuously differentiable function such that

(a). the set $S_c = \{\mathbf{x} : V(\mathbf{x}) \leq c\}$ is bounded for some c

(b). $\dot{V}(\mathbf{x}) \leq 0$ whenever $\mathbf{x} \in S_c$

then S_c is a positively invariant set

LaSalle's Principle: Define the following two sets

$$E = \{\mathbf{x} \in S_c : \dot{V}(\mathbf{x}) = 0\}$$
$$M = \{\text{union of all positively invariant sets in } E\}$$

then every trajectory starting in S_c tends to M as $t \to \infty$

LaSalle's principle: application to Duffing oscillator

Governing system:

$$\dot{x} = y$$
$$\dot{y} = x - x^3 - \gamma y, \qquad \gamma > 0$$

- Let $V(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$
  then $\dot{V}(x,y) = -\gamma y^2$

- Let $S_c = \{(x,y) : V(x,y) \leq c\}$ for any chosen $c > 0$,
  then S_c is positively invariant since $\dot{V} \leq 0$

- Here $E = \{(x,y) : y = 0\}$ and $M = \{(-1,0), (0,0), (1,0)\}$

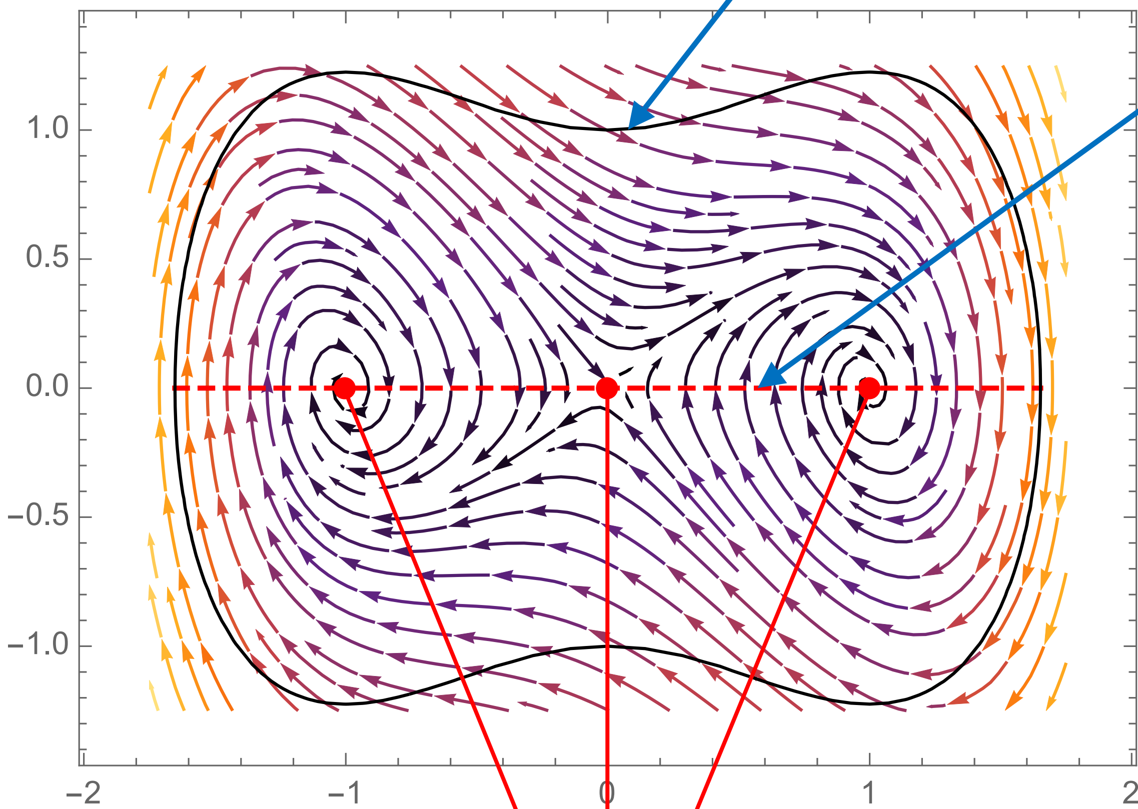
- LaSalle's principle says all trajectories in S_c ultimately converge to M, and hence to one of the three equilibria

LaSalle and Duffing: visualization

Phase plane for $\gamma = 0.5$

$S_{0.5} = \{(x, y) : V(x, y) = 0.5\}$

$E = \{(x, y) : y = 0\}$



$M = \{(-1,0), (0,0), (1,0)\}$

Characterization of orbits

We aim to classify the types of attractors in the **phase plane**.
First some definitions:

- A **homoclinic orbit** is a trajectory that joins a saddle point equilibrium to itself – it moves away from the equilibrium on an unstable manifold and returns on a stable manifold

- A **heteroclinic orbit** is a trajectory that joins two distinct equilibrium points

- A **separatrix cycle** partitions the phase plane into two regions with different characteristics – there are many ways to construct separatrix cycles

Homoclinic orbit example

Consider the Hamiltonian system

$$\dot{x} = y$$
$$\dot{y} = x + x^2$$
$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 - \frac{1}{3}x^3$$

- Solution trajectories are the level sets of the Hamiltonian function (constant energy curves in phase space):

$$\Gamma_c = \left\{(x, y)\colon\ y^2 - x^2 - \frac{2}{3}x^3 = c\right\}$$

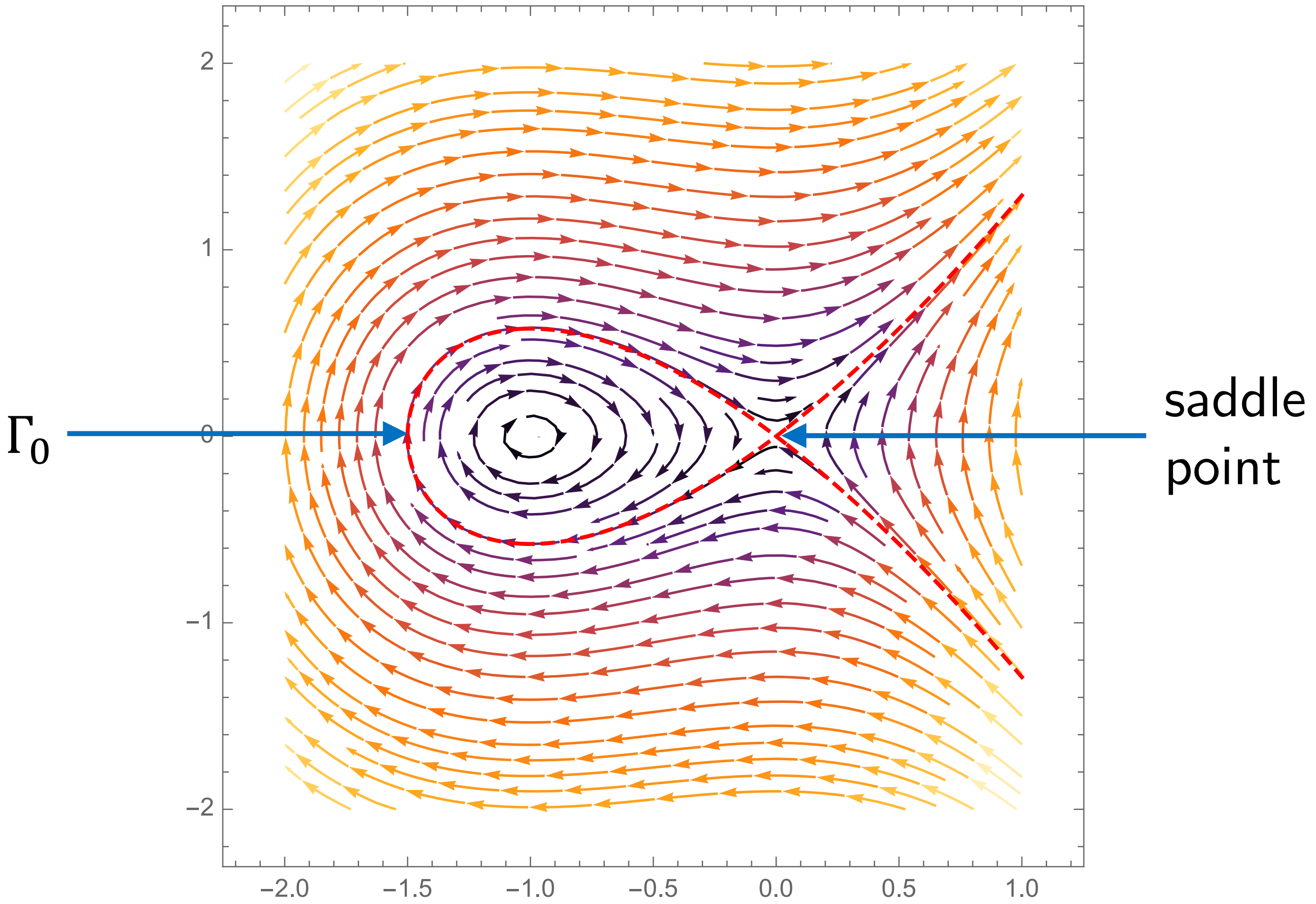
- For $c = 0$, $\Gamma_0 = \left\{(x, y)\colon\ y^2 = x^2 + \frac{2}{3}x^3\right\}$ is a trajectory that passes through a saddle point at the origin:

$$D\mathbf{f}(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda = \{1, -1\}$$

Homoclinic orbit visualization

Phase portrait of $\dot{x} = y$

$\dot{y} = x + x^2$



Γ_0 contains a homoclinic orbit

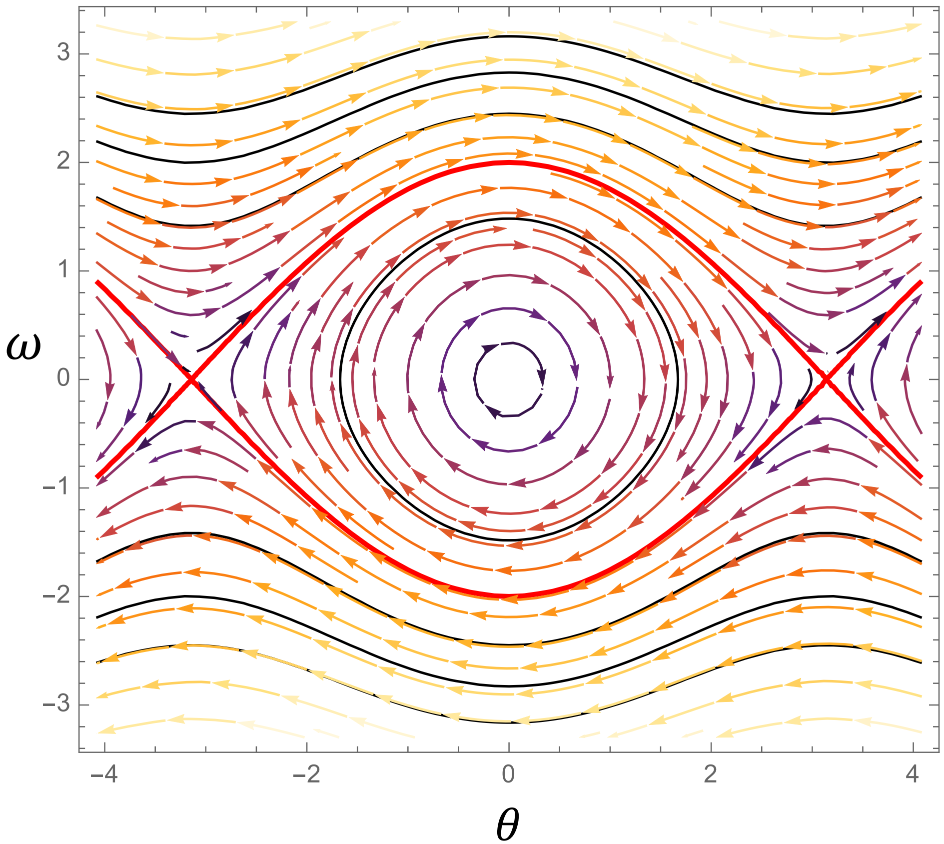
Heteroclinic orbit example

The phase plane of the undamped pendulum has heteroclinic orbits

- There are saddle-point equilibria whenever the pendulum points upward ($\theta = \pi$); a heteroclinic orbit connects these

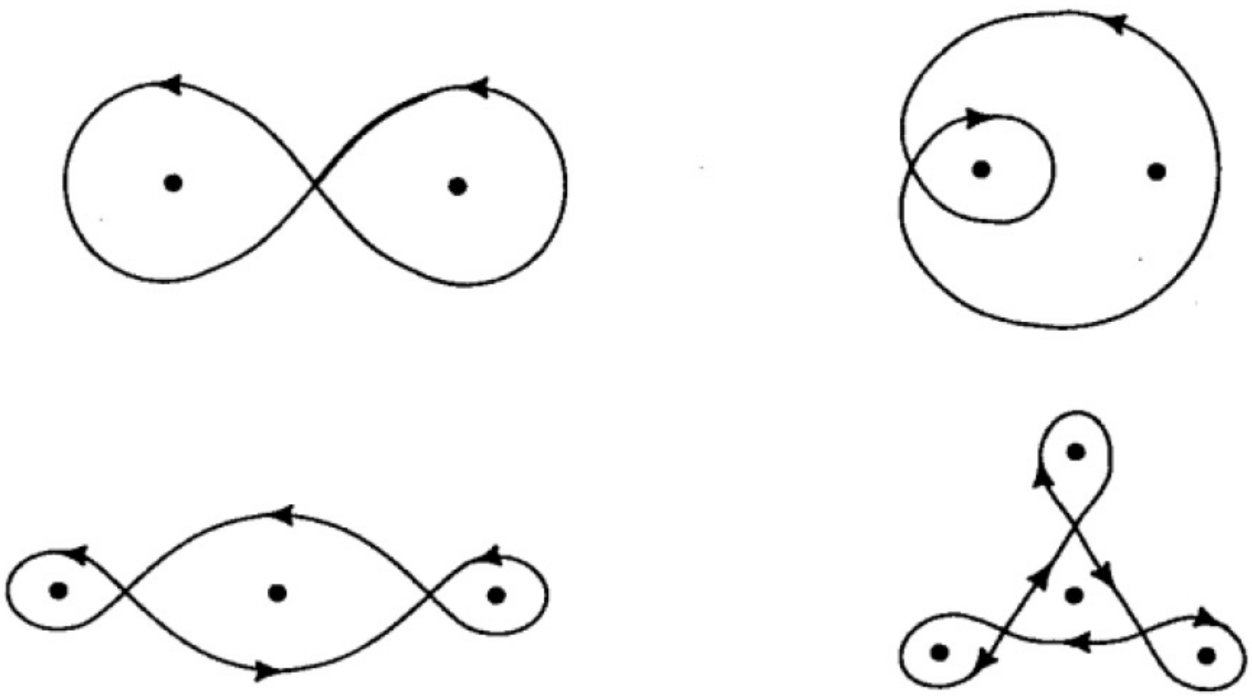
- The two heteroclinic cycles in the upper and lower half-plane define a heteroclinic separatrix cycle (in red)

Phase portrait of $\dot{\theta} = \omega$
$\dot{\omega} = -\sin\theta$



Compound separatrix cycles

- A compound separatrix cycle is an orbit that surrounds multiple equilibrium points with compatibly orientated separatrices



- Note that everywhere such a cycle crosses itself must also be an equilibrium point

Poincaré-Bendixson theorem in the plane

Theorem: Let M be a positively invariant region of a system with a 2-d phase space containing only a finite number of equilibria. Let $\mathbf{x} \in M$ and consider $\omega(\mathbf{x})$. One of the following possibilities must hold:

i. $\omega(\mathbf{x})$ is an equilibrium

ii. $\omega(\mathbf{x})$ is a closed orbit

iii. $\omega(\mathbf{x})$ consists of a finite number of equilibria $\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*$ and orbits Γ with $\alpha(\Gamma) = \mathbf{x}_i^*$ and $\omega(\Gamma) = \mathbf{x}_j^*$

Case iii defines a set of heteroclinic orbits (e.g. undamped pendulum)

Observations:

- If all the equilibria in M are stable, then there can be only one
- If there are no stable equilibria in M, then M must contain a closed orbit (stable limit cycle)

Poincaré-Bendixson and the Duffing oscillator

Back to the Duffing Oscillator:

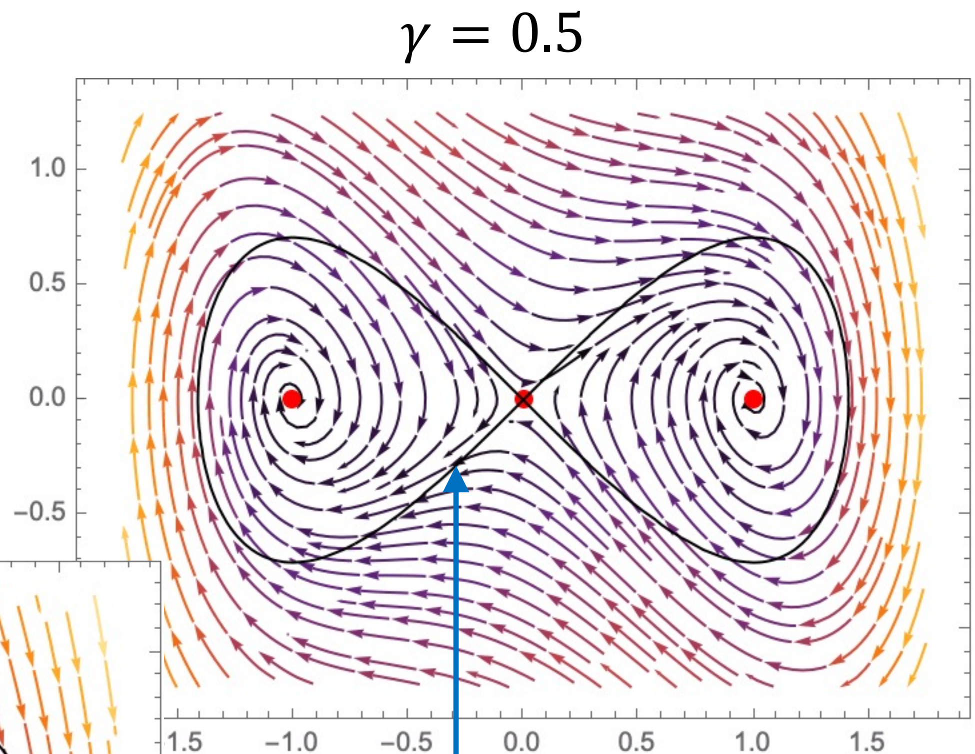
$$\dot{x} = y$$
$$\dot{y} = x - x^3 - \gamma y, \qquad \gamma > 0$$

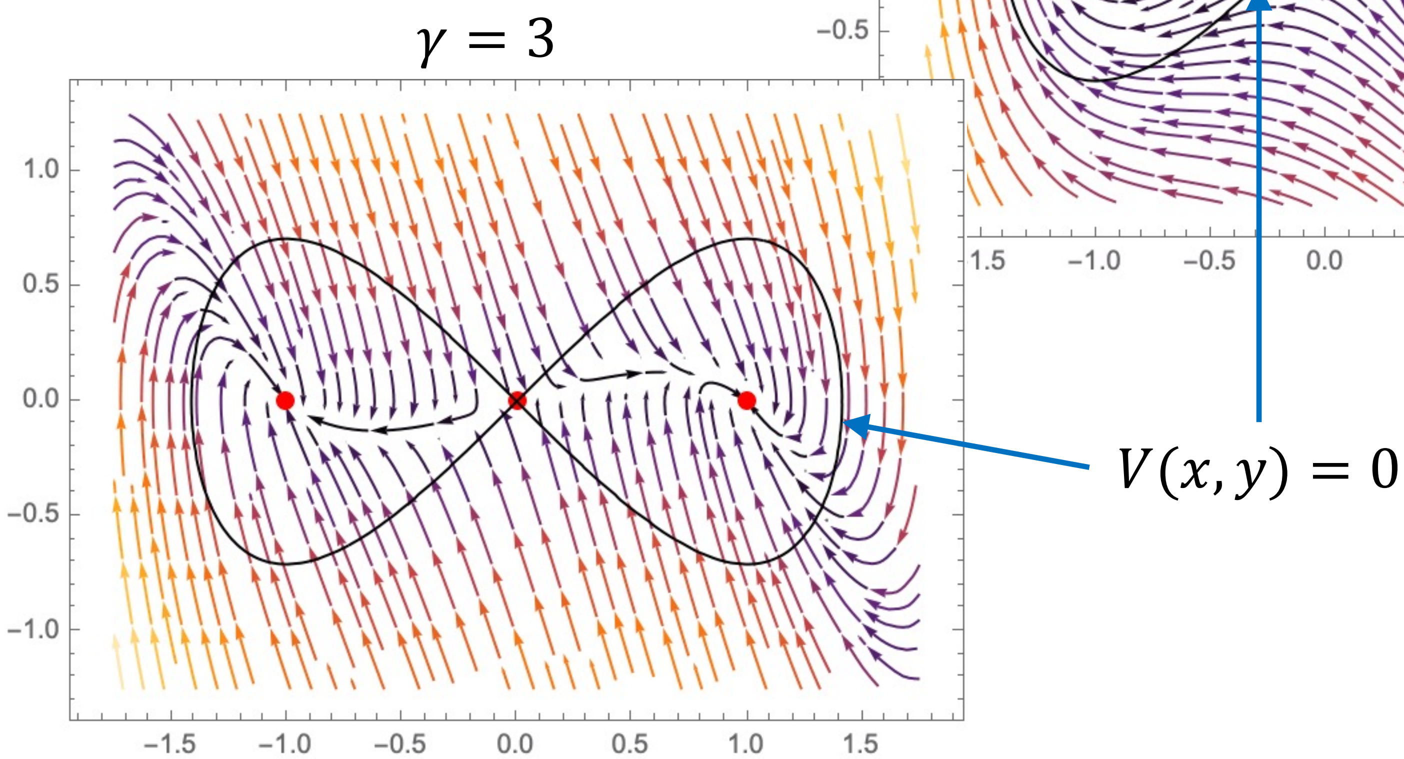
- Level sets of $V(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$ are positively invariant since $\dot{V}(x,y) = -\gamma y^2$
- Three equilibria lie in $S_c = \{(x,y) : V(x,y) \leq c\}$ for $c > 0$:
  - an unstable equilibrium $(0,0)$
  - two stable equilibria $(-1,0), (1,0)$
- For $c = 0$, S_0 splits into two sets that share a common point at $(0,0)$
- Therefore all trajectories leaving the unstable equilibrium point must end up at one of the stable equilibria

Two types of behaviour for the Duffing oscillator

ω limits are:

- foci for $0 < \gamma \leq \sqrt{8}$
- nodes for $\gamma > \sqrt{8}$





Additional Poincaré-Bendixson example

Consider the autonomous system:

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

- Linearizing around the **only** equilibrium point (at $(0,0)$) gives Jacobian

$$D\mathbf{f}(0,0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \;\Rightarrow\; \lambda = 1 \pm j \quad \text{(an unstable spiral)}$$

- If $V = x_1^2 + x_2^2$, then $\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1^2 + 2x_2^2 - 2(x_1^2 + x_2^2)^2$

$$\Rightarrow \quad \dot{V} = 2V(1 - V)$$

$$\Rightarrow \quad \dot{V} \leq 0 \text{ whenever } V \geq 1$$

  so $S_c = \{(x, y) : V \leq c\}$ is invariant for $c \geq 1$

- Since there is only an unstable equilibrium point inside S_1, Poincaré-Bendixson implies that S_1 must contain a stable limit cycle

Visualizing the additional example

The phase portrait has an unstable spiral within a stable limit cycle

$x_1^2 + x_2^2 = 1$