Lecture 6: Limit cycles

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Lecture 6 overview

- This lecture will focus on analyzing **limit cycles**, conditions for their existence and stability
- Last lecture the Poincaré-Bendixson theorem gave us criteria to establish whether closed orbits exist; we can also establish if they do not exist, through **Bendixson's** and **Dulac's criteria**
- **Index theory** will help us characterize closed trajectories in the phase plane, and to determine whether it is possible for orbiting trajectories to exist
- We will assign stability to limit cycles through the concept of a **Poincaré map** to help us analyze them

Periodicity

Definition: A solution $\phi(\mathbf{x}_0, t)$ of the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ satisfying $\phi(\mathbf{x}_0, 0) = \mathbf{x}_0$ is called **periodic** if there exists some $T > 0$ such that $\phi(t, \mathbf{x}_0) = \phi(t + T, \mathbf{x}_0)$ for all $t \in \mathbb{R}$

- Given a periodic solution $\phi(\mathbf{x}_0, t)$, the minimal value of $T > 0$ for which $\phi(t, \mathbf{x}_0) = \phi(t + T, \mathbf{x}_0)$ is called the **period** of the solution

- This lecture only considers orbits with finite period

  Hence we exclude separatrix cycles because it takes infinite time for homo-/heteroclinic connections to go from α to ω limits, so $t + T$ then makes no sense

Proving a periodic orbit does not exist

- Bendixson's criterion can be used to show that a given 2nd order dynamical system does not have any periodic solutions

- Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ denote the state of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and recall that $\mathrm{div}(\mathbf{f}) = \nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}$ if

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$$

Bendixson's criterion: if $\nabla \cdot \mathbf{f}$ is not identically zero, and if $\nabla \cdot \mathbf{f}$ does not change sign in a simply connected region D of the phase plane, then the 2nd order system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no closed orbits in D

Outline proof for Bendixson's criterion

- Since x and y are parametric in t, the solution trajectories satisfy

$$\begin{aligned}\dot{x} &= f_x \\ \dot{y} &= f_y\end{aligned} \quad \Longrightarrow \quad \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{f_y}{f_x} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{f_y}{f_x}$$

- Suppose a closed orbit $\Gamma \subset D$ exists,
  then $f_x\,dy - f_y\,dx = 0$ on Γ so

$$\oint_\Gamma (f_x\,dy - f_y\,dx) = 0$$

  and by Stokes's theorem, if Γ encloses a region $S \subset D$ then

$$\oint_\Gamma (f_x\,dy - f_y\,dx) = \int_S (\nabla\cdot\mathbf{f})\,dx\,dy = \int_S \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) dx\,dy = 0$$

- So If $\nabla \cdot \mathbf{f}$ is nonzero and doesn't change sign in D, then our supposition must be false, i.e. no orbit is possible

Modification: Dulac's criterion

Consider the same differential equations, but also allow the functions $f_x(x, y)$ and $f_y(x, y)$ to be multiplied by another function $B(x, y)$

Dulac's criterion: if B is a continuously differentiable function on a domain D of the phase plane, and if the quantity

$$\frac{\partial(Bf_x)}{\partial x} + \frac{\partial(Bf_y)}{\partial y} = \nabla \cdot (B\mathbf{f})$$

is not identically zero and does not change sign in the domain, then the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no closed orbits in the domain D

Bendixson example 1

Return again to the Duffing oscillator, which is described for $\gamma \geq 0$ by

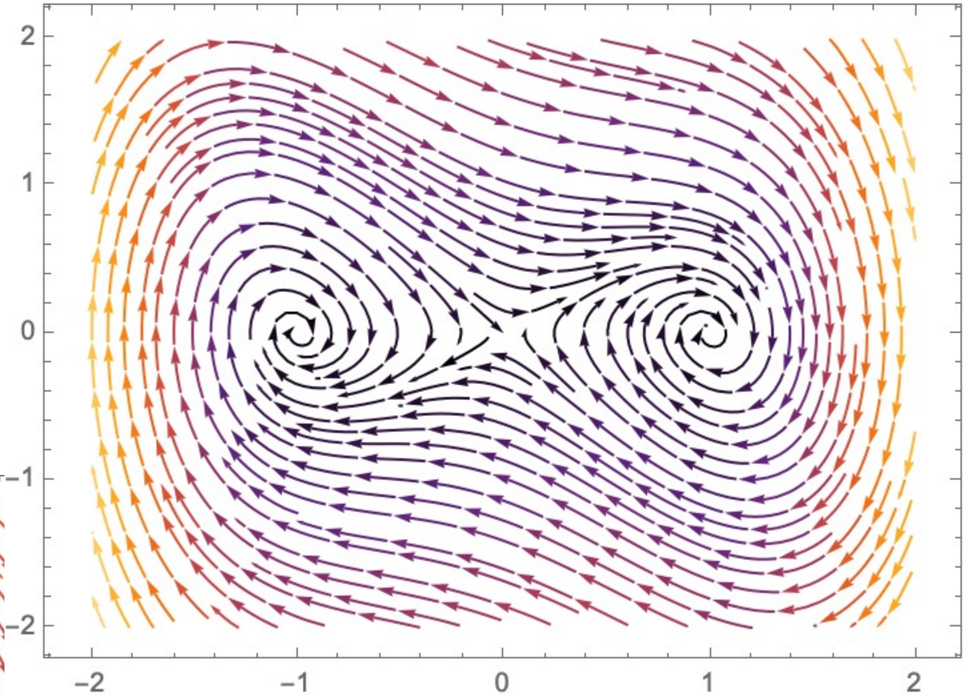
$$\begin{aligned} \frac{dx}{dt} &= y &&= f_x(x, y) \\ \frac{dy}{dt} &= x - x^3 - \gamma y &&= f_y(x, y) \end{aligned}$$

- Here $\nabla \cdot \mathbf{f} = -\gamma$, so Bendixson's criterion implies that:
  - for $\gamma \neq 0$ there are no solution trajectories that are closed orbits
  - for $\gamma = 0$ periodic solutions are possible
- As we saw in lecture 4, for $\gamma = 0$ the system is Hamiltonian, and its trajectories can be studied using the level sets of the Hamiltonian function

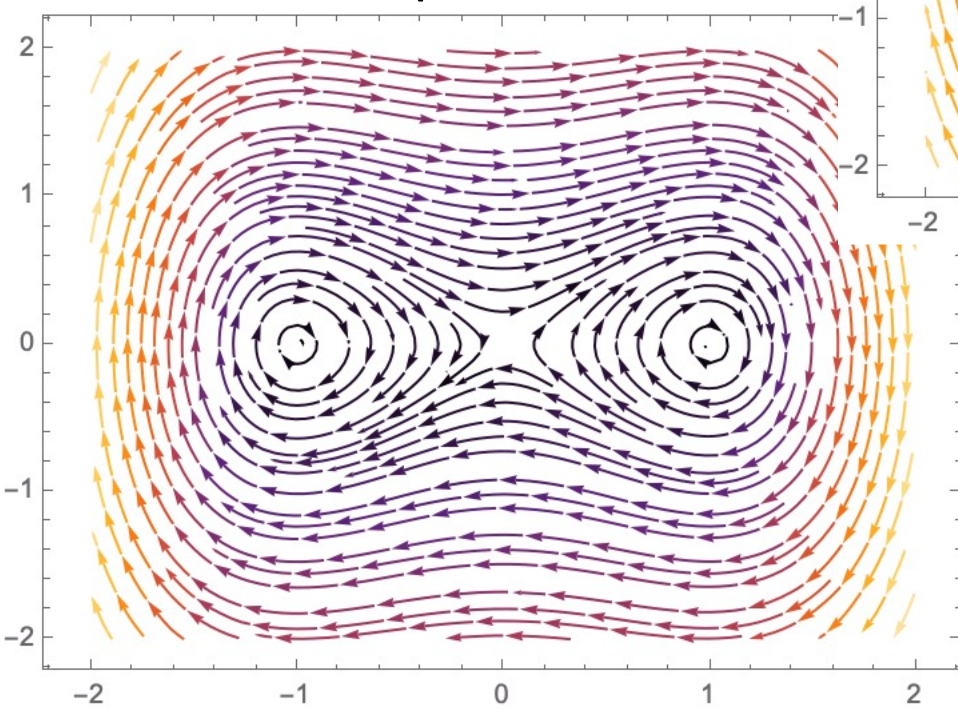
Example 1 visualisation

- no closed orbits for $\gamma > 0$
- closed orbits for $\gamma = 0$

$\gamma = 0.5$



$\gamma = 0$



Bendixson example 2

Now modify the second Duffing oscillator equation to get

$$\frac{dx}{dt} = y = f_x(x, y)$$
$$\frac{dy}{dt} = x - x^3 - \gamma y + x^2 y = f_y(x, y)$$

- Here $\nabla \cdot \mathbf{f} = -\gamma + x^2$
- Using Bendixson's criterion, it be can't concluded that there are no closed orbits
  - there can't be a closed orbit entirely within a region of phase space where $\nabla \cdot \mathbf{f} < 0$ or $\nabla \cdot \mathbf{f} > 0$
  - but orbits could exist because $\nabla \cdot \mathbf{f}$ can change sign

Gradient systems and orbits

Recall that for a gradient system we have $\dot{\mathbf{x}} = -\nabla V$

- Consider the time-derivative of the potential function:

$$\dot{V} = \frac{dV}{dt} = \nabla V \cdot \dot{\mathbf{x}} = -\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = -\|\dot{\mathbf{x}}\|^2$$

- If the solution is on a closed orbit of period T, then we must have

$$V\big(\mathbf{x}(T+t)\big) - V\big(\mathbf{x}(t)\big) = 0 \quad \forall t$$

But integrating $\dot{V}$ w.r.t. t gives

$$V\big(\mathbf{x}(T+t)\big) - V\big(\mathbf{x}(t)\big) = \int_t^{t+T} \dot{V}\, dt = -\int_t^{t+T} \|\dot{\mathbf{x}}\|^2\, dt$$

and the only way this can equal zero is if $\mathbf{x}(t)$ is at an equilibrium point, so **gradient systems cannot have periodic solutions**

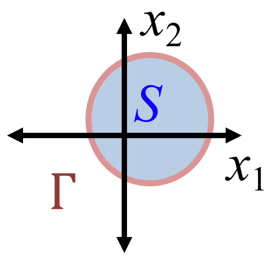
Gradient systems and orbits

Index theory

- For two-dimensional systems, we have seen that analyzing solution trajectories is facilitated by using techniques applicable to fluid flow

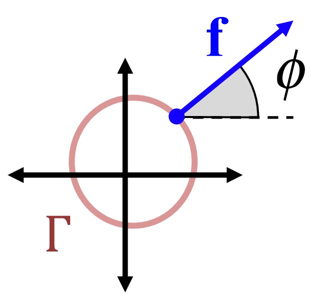
- Bendixson's criterion checks the circulation of a vector field:

$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}\right\} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$



$$\oint_\Gamma \begin{bmatrix} -f_2 \\ f_1 \end{bmatrix} \cdot d\mathbf{l} = \int_S \nabla \times \begin{bmatrix} -f_2 \\ f_1 \end{bmatrix} \cdot d\mathbf{S} = \int_S \nabla \cdot \mathbf{f}\, dS$$

- Index theory translates circulation into a quantity that takes simple integer values; it quantifies the net change in the angle a flow makes with the x_1 axis when traversing loop ∂S



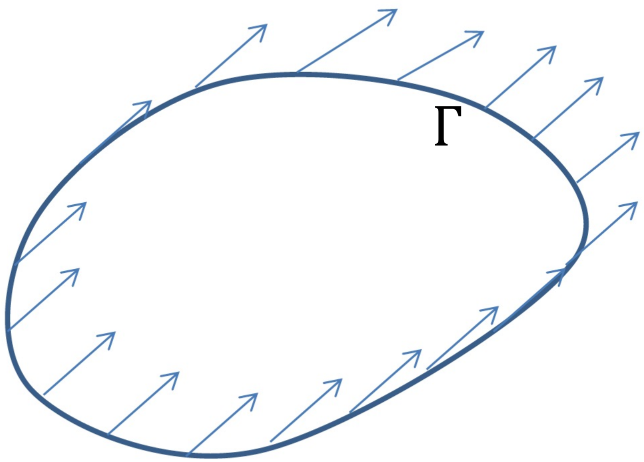
$$\phi(x_1, x_2) = \arctan\left(\frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}\right)$$

Index of a curve

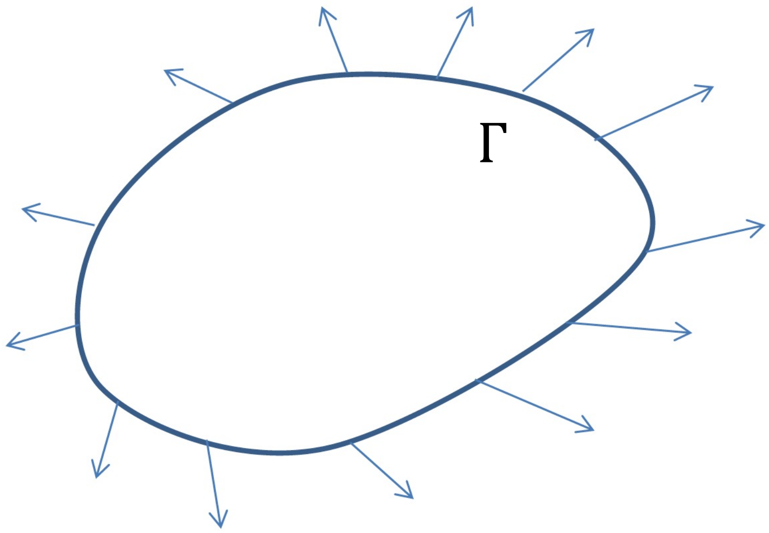
- The index of a non-intersecting, continuous differentiable closed plane curve Γ (i.e. a simple loop), written $I(\Gamma)$, is defined as

$$I(\Gamma) = \frac{1}{2\pi} \oint_\Gamma d\phi$$

- Qualitatively, the index measures how many times the vectors on the curve rotate anticlockwise during one anticlockwise trip around the loop



index = 0



index = 1

Properties of indices

- The index is always an integer (one must always rotate by a multiple of 2π to get the flow angle back to where it started)
- If there are no equilibria inside a loop Γ, then its index is $I(\Gamma) = 0$
- If loop Γ coincides with a closed orbit, then $I(\Gamma) = 1$
- If loop Γ encloses a saddle equilibrium point, then $I(\Gamma) = -1$
- If loop Γ encloses any other equilibrium point, then $I(\Gamma) = 1$
- The index of a loop that encloses multiple equilibria is the sum of the indices of loops around the individual equilibria enclosed

General conclusions from indices

- Any loop of index 0 that does not contain equilibrium points cannot be a solution trajectory

  To be a valid trajectory, it would have to be an orbit, but that requires it to have index 1, not 0

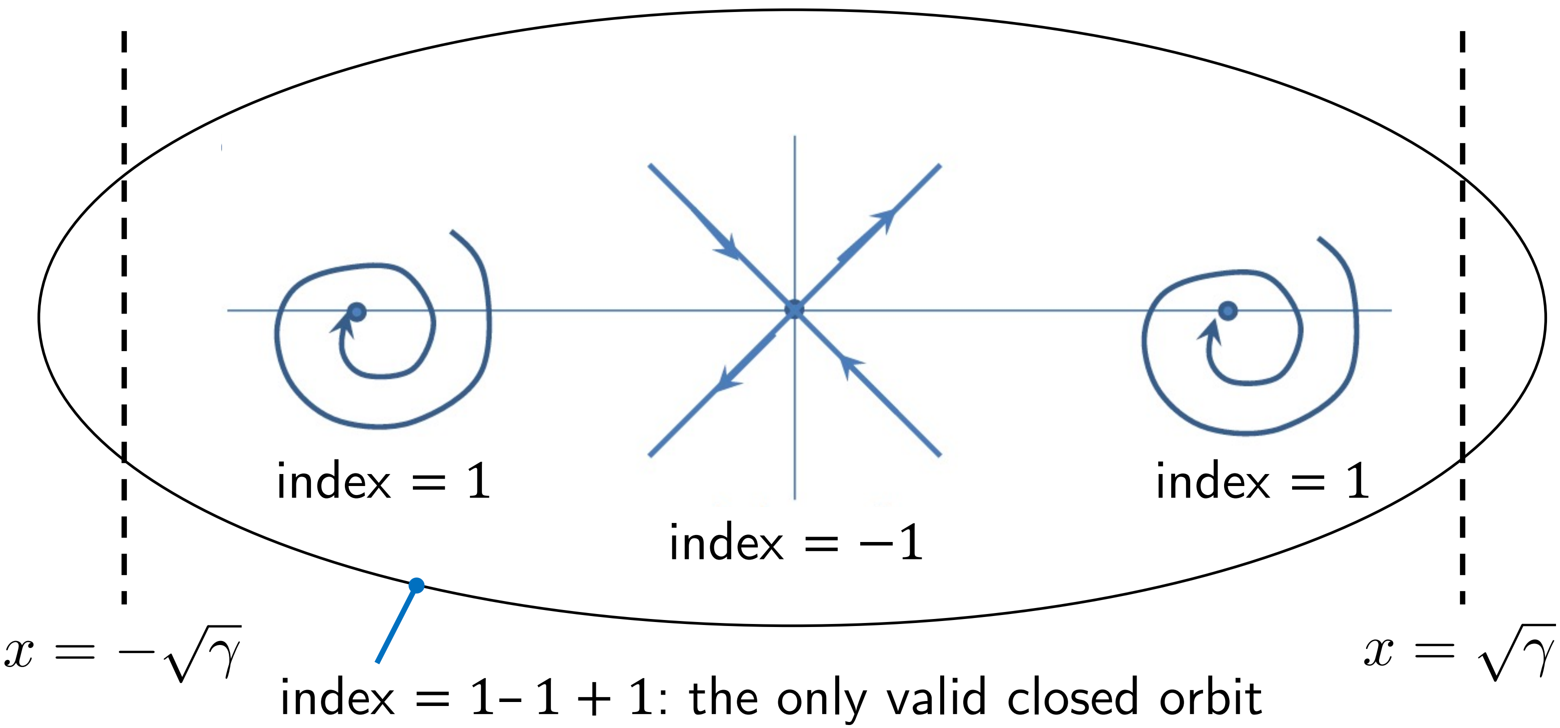
- Any loop around a single saddle node cannot be a solution trajectory

  To be a valid trajectory, it would have to be an orbit, but that requires it to have index 1, not -1

Return to Bendixson example 2

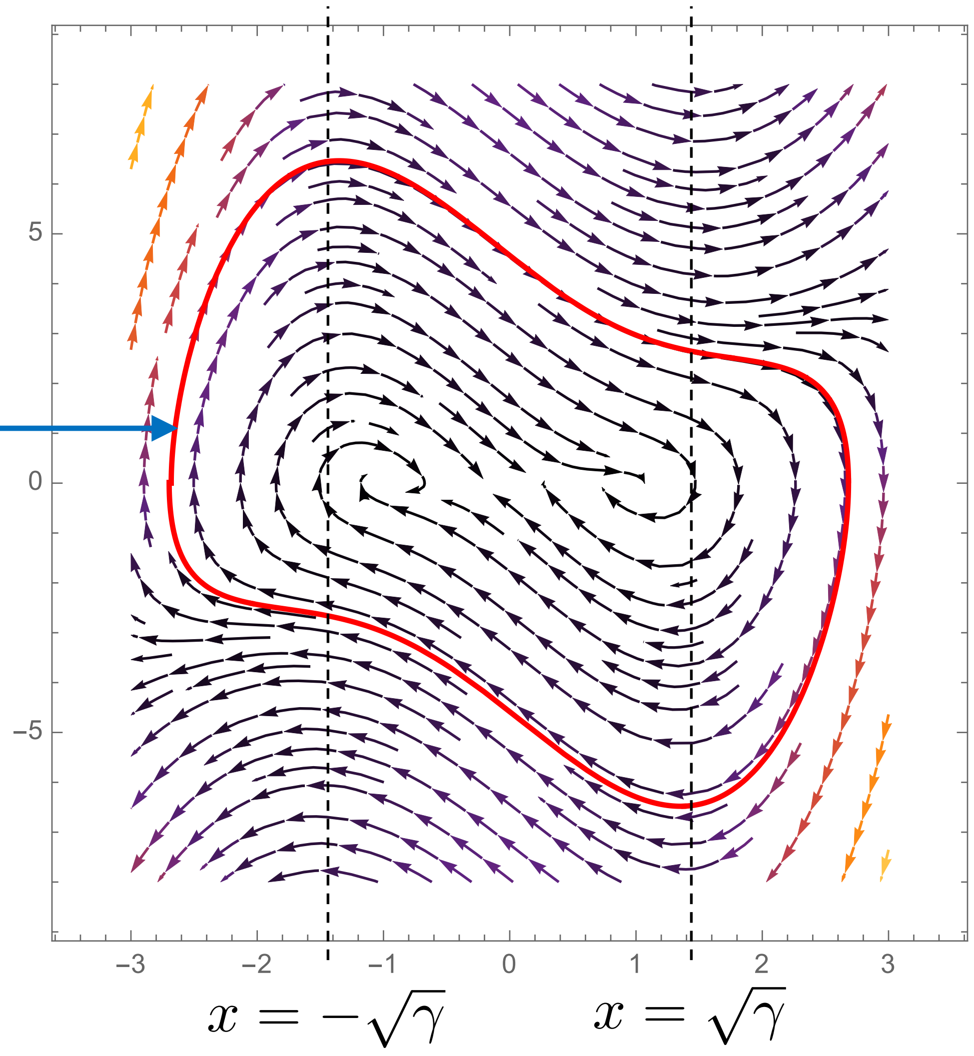
- Governing system: $$\dot{x} = y$$ $$\dot{y} = x - x^3 - \gamma y + x^2 y$$

- Three hyperbolic equilibria: $(-1,0), (0,0), (1,0)$
  stable nodes or foci at $(\pm 1, 0)$ and a saddle node at $(0,0)$



Example 2 visualisation

Phase portrait for $\gamma = 2$



Periodic solution in predicted location!

Another index theory example

Consider
$$\dot{x}_1 = x_1(3 - x_1 - 2x_2)$$
$$\dot{x}_2 = x_2(2 - x_1 - x_2)$$

- Equilibrium points: $\mathbf{x}^* = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

- Jacobian: $D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 3 - 2x_1 - 2x_2 & -2x_1 \\ -x_2 & 2 - x_1 - 2x_2 \end{bmatrix}$

$$\det(D\mathbf{f}(\mathbf{x}^*) - \lambda I) = 0$$
$$\implies \lambda = (3, 2), (-2, -1), (-3, -1), (-1 \pm \sqrt{2})$$

- properties: unstable node, stable node, stable node, saddle

  indices: $1,\ 1,\ 1,\ -1$

Continuing the example

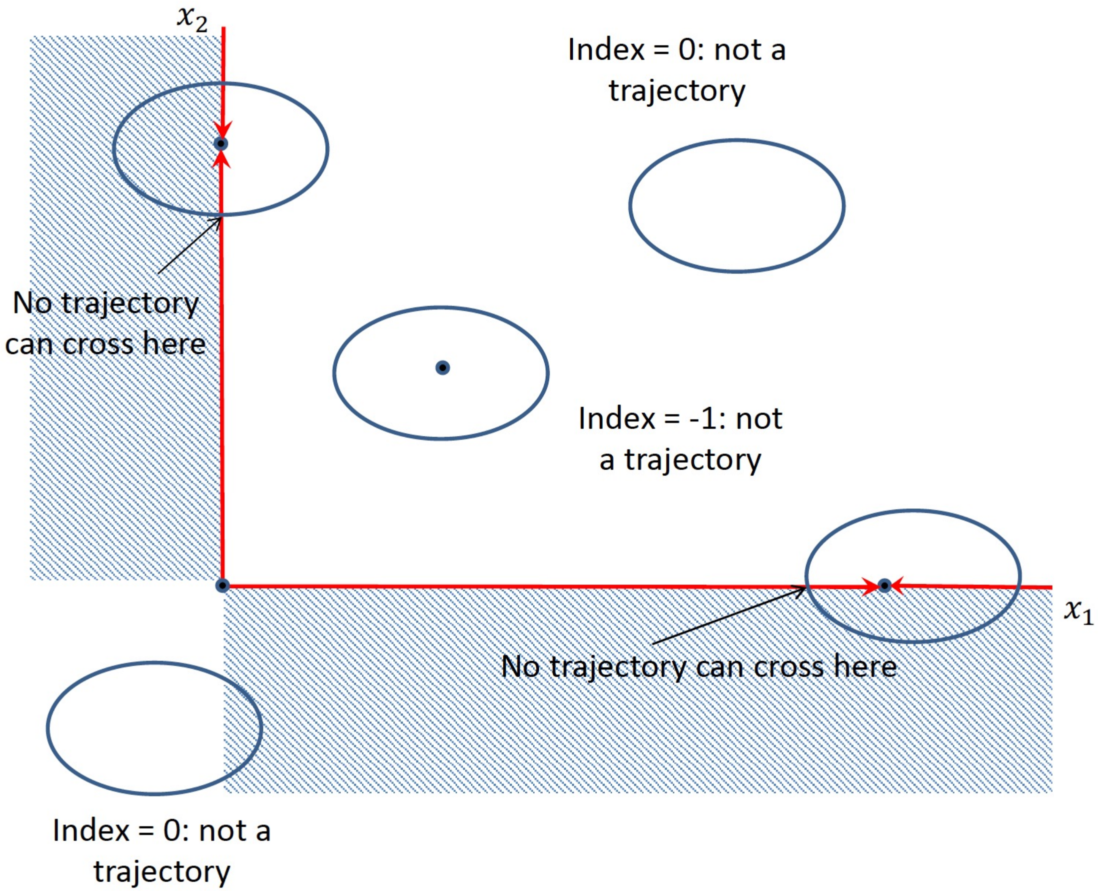
Equilibrium points $\mathbf{x}^* = \left\{ \begin{matrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 2 \end{bmatrix}, & \begin{bmatrix} 3 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ I = 1 & I = 1 & I = 1 & I = -1 \end{matrix} \right\}$

- a valid orbit must have index $I = 1$
- trajectories cannot cross
- no equilibria in the 2nd, 3rd, or 4th quadrants so they cannot contain a closed trajectory
- there are trajectories lying on the x_1 and x_2 axes, so no trajectory can cross into the 2nd, 3rd, or 4th quadrants
- since trajectories cannot encircle the equilibria that lie on the axes, it is not possible enclose a set of indices that add to 1

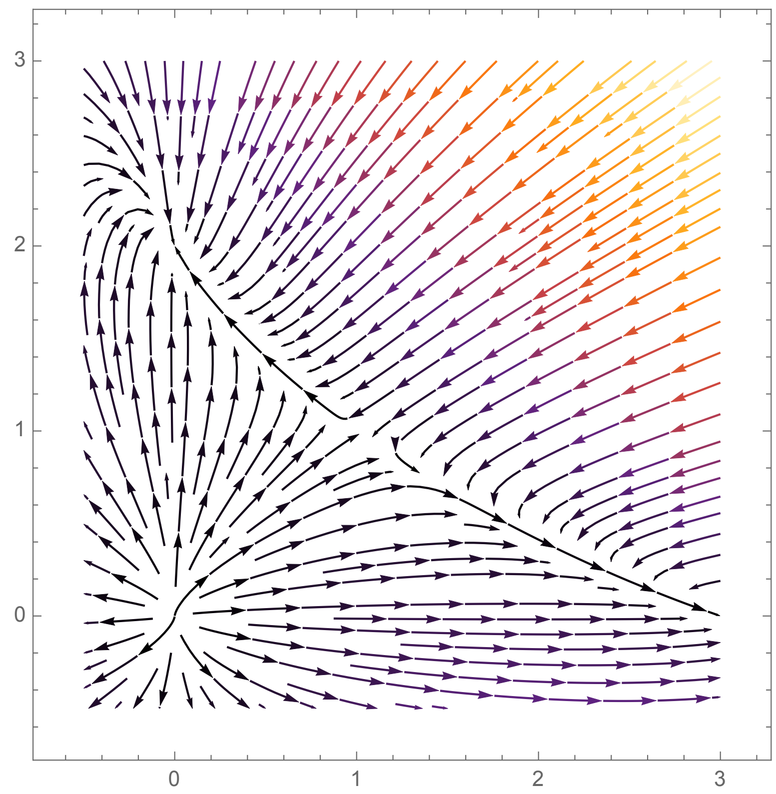
$\implies$ there are no possible closed orbits

Visualization

- Graphical illustration of arguments



- Phase portrait



Limit cycles and stability

So far we have used the term limit cycle informally but it is worth putting some rigour behind our terms

- Limit cycles are isolated periodic orbits, which can be stable or unstable (a cycle around a linear centre is not isolated and hence is not a limit cycle)

- In the phase plane, a limit cycle is necessarily the α or ω limit set of some trajectory other than itself

Definition: A periodic orbit Γ is said to be **stable** if for every $\epsilon > 0$ there is a neighbourhood U of Γ such that for $\mathbf{x} \in U$ the distance between $\phi(t, \mathbf{x})$ and Γ is less than ϵ. Orbit Γ is called **asymptotically stable** if it is stable and, for all $\mathbf{x} \in U$, this distance tends to zero as t tends to infinity

Conditions for limit-cycle stability

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ have a periodic solution $\mathbf{x} = \gamma(t), 0 \leq t \leq T$, then the periodic orbit Γ lies on $\gamma(t)$

The periodic orbit is asymptotically stable only if

$$\int_0^T \nabla \cdot \mathbf{f}(\gamma(t))\, dt \leq 0$$

- For planar systems, if Γ is the ω limit set of all trajectories in the neighbourhood of Γ, then it is a **stable** limit cycle
- For planar systems, if Γ is the α limit set of all trajectories in the neighbourhood of Γ, then it is an **unstable** limit cycle
- For planar systems, if Γ is the ω limit set for one trajectory and the α limit set for another, it is a **semi-stable** limit cycle

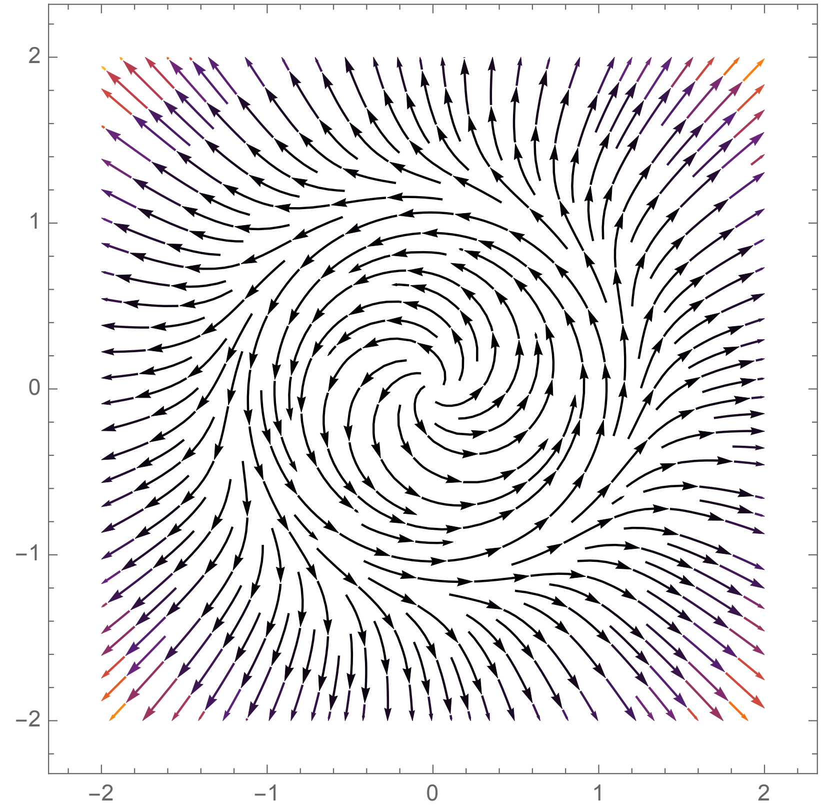
Limit cycle example

Examine the autonomous system

$$\dot{x} = -y + x(1 - x^2 - y^2)^2 \qquad \qquad \dot{r} = r(1 - r^2)^2$$
$$\Longleftrightarrow$$
$$\dot{y} = x + y(1 - x^2 - y^2)^2 \qquad \qquad \dot{\theta} = 1$$

- For $r \neq 1$, $\dot{r} > 0$
  therefore solution trajectories
  spiral outwards

- For $r = 1$, $\dot{r} = 0$
  therefore a semi-stable
  limit cycle



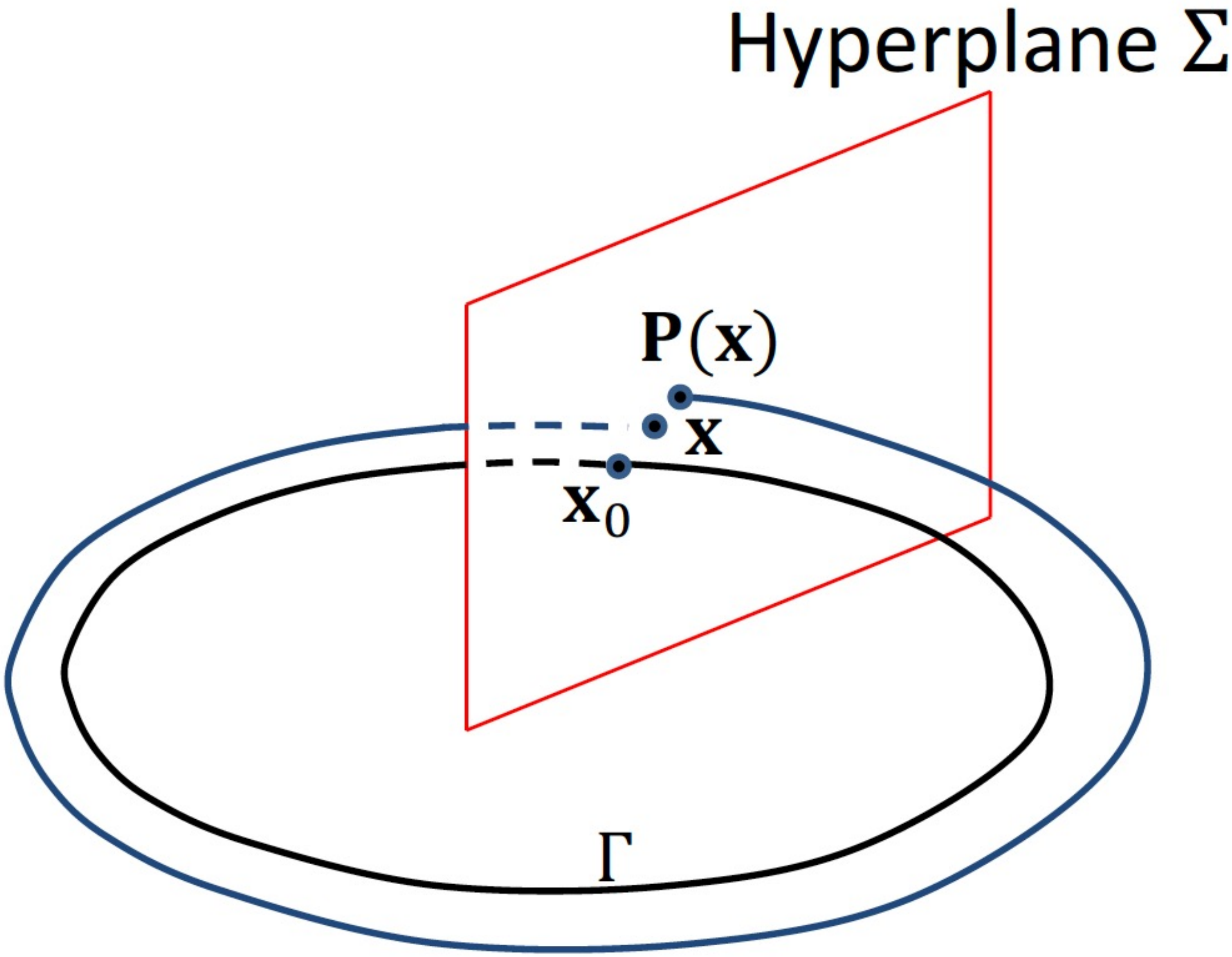
The Poincaré map

- The Poincaré map (sometimes called the 'return map') is an important tool for the analysis of dynamical systems
- For a periodic orbit, consider a hyperplane Σ that is perpendicular to the orbit's trajectory
- Given a point $\mathbf{x}$ on the orbit and in the hyperplane, consider where the point moves to once it has traversed the orbit once; this process defines a map

$$\mathbf{x} \mapsto \mathbf{P}(\mathbf{x})$$

- As this mapping is iterated, the intersection point moves in the perpendicular hyperplane
- If it is a periodic orbit, then the iteration of the map will arrive at a stationary point, $\mathbf{x} = \mathbf{P}(\mathbf{x})$

Graphical representation



Poincaré map example

Return to the system we discussed in Lecture 5:

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

- In Lecture 5 we showed that this has a stable limit cycle, which is an attractor for the whole plane excluding the origin
- Solving for $\big(r(t), \theta(t)\big)$ given initial condition (r_0, θ_0):

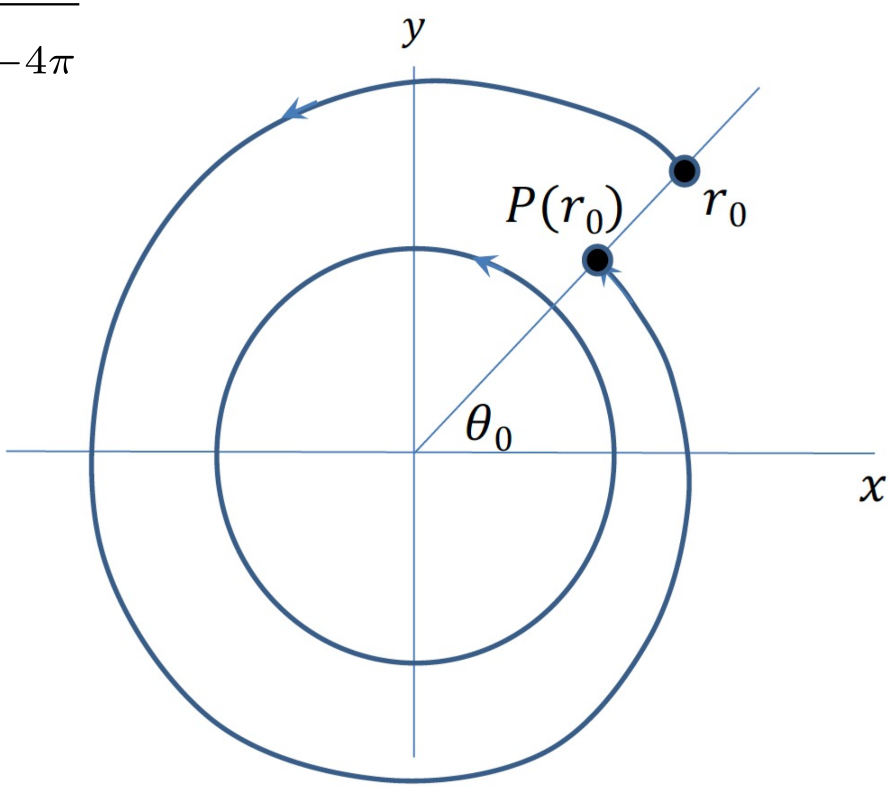
$$\frac{dr}{dt} = r(1 - r^2) \quad \Longrightarrow \quad \int_{r_0}^{r(t)} \frac{dr}{r(1 - r^2)} = \int_0^t dt = t$$

$$\Longrightarrow \quad r(t) = \frac{1}{\sqrt{1 - (1 - \frac{1}{r_0^2})e^{-2t}}}, \quad \theta = \theta_0 + t$$

Poincaré map example continued

- Consider the hyperplane Σ defined by the ray $\theta = \theta_0$ through the origin that's crossed by a solution trajectory at times $t = 0, 2\pi, 4\pi, \ldots$

$$P(r_0) = \frac{1}{\sqrt{1 - \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}}}$$



- Here $P(1) = 1$, so $r = 1$ is a fixed point

- This is a stable limit cycle because

$$\left.\frac{dP}{dr}\right|_{r=1} = e^{-4\pi} < 1$$