# Lecture 8: Chaos

Mark Cannon
mark.cannon@eng.ox.ac.uk

# Lecture 8 overview

- Now we shift attention away from differential equations and consider discrete time systems, particularly 1-D maps:

$$x_{k+1} = F(x_k)$$

- We will investigate the behaviour of these maps graphically using cobweb plots

- Unlike 1-D differential systems, the discrete 1-D map can exhibit the phenomenon of chaos; a few mathematical techniques will help analyze such systems

- Finally we return to differential systems, examining the chaotic Lorenz equations and identifying a strange attractor

# Analysis of map fixed points

- Given a discrete map $\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k)$, a fixed point $\mathbf{x}^*$ satisfies

$$\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$$

- The map's behaviour near the equilibrium point $\mathbf{x}^*$ can be characterised by linearising $\mathbf{F}$

$$\begin{aligned} \mathbf{x}^* + \mathbf{w}_{k+1} &= \mathbf{F}(\mathbf{x}^*) + D\mathbf{F}(\mathbf{x}^*)\mathbf{w}_k + \dots \\ \implies \qquad \mathbf{w}_{k+1} &= D\mathbf{F}(\mathbf{x}^*)\mathbf{w}_k + \dots \end{aligned}$$

- Stability can be assessed by analysing properties of the Jacobian $D\mathbf{F}$ at the equilibrium point

- Note that for the 1-D case the Jacobian of the map is simply

$$DF(x^*) = \left.\frac{dF}{dx}\right|_{x^*}$$

# Stability of fixed points

- For a linear map the stability of the fixed point at the origin can be understood by diagonalisation (cf example sheet 1)

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \quad \Longrightarrow \quad \mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0 \quad \Longrightarrow \quad \mathbf{x}_n = \mathbf{V}\boldsymbol{\Lambda}^n\mathbf{V}^{-1}\mathbf{x}_0$$

- The behaviour as $n \to \infty$ determines the stability of the map, e.g. in 2-D

$$\lim_{n\to\infty} \boldsymbol{\Lambda}^n = \lim_{n\to\infty} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \qquad \begin{array}{lcl} |\lambda|, |\mu| > 1 & \Longrightarrow & \text{unstable node} \\ |\lambda|, |\mu| < 1 & \Longrightarrow & \text{stable node} \\ |\lambda| > 1,\ |\mu| < 1 & \Longrightarrow & \text{saddle} \end{array}$$

- Eigenvalues inside the unit circle are stable

- Hence for the 1-D case $x^*$ is a stable fixed point if $|DF(x^*)| < 1$

# Cobweb diagrams

- Cobweb diagrams represent graphically the iterates of a map

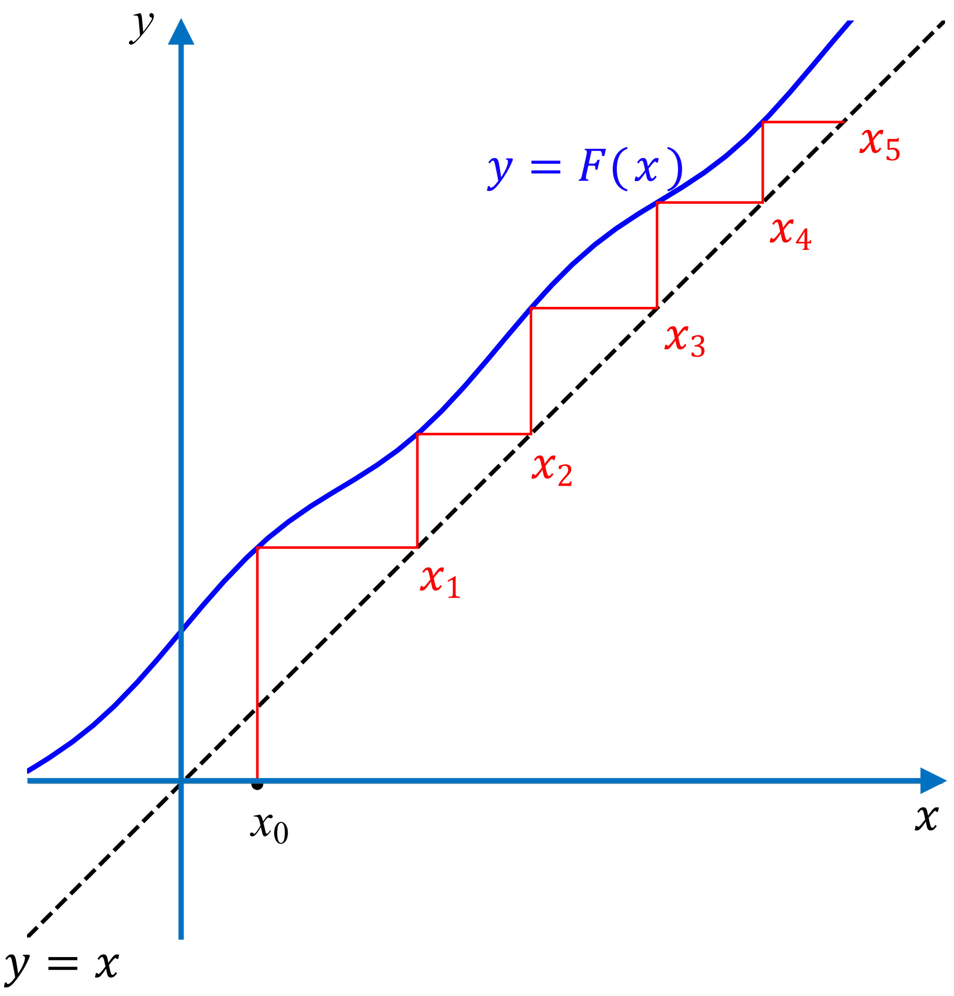
- Given map $x_{k+1} = F(x_k)$, use the following procedure:

  1. Go to position $x_k$ on the horizontal axis

  2. Draw a vertical line up to $y = F(x_k)$

  3. Draw a horizontal to $y = x$ and set $x_{k+1} = F(x_k)$

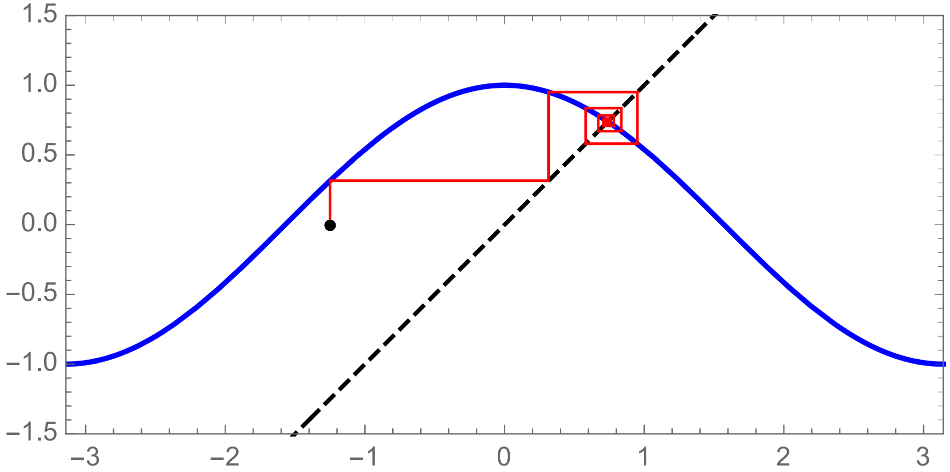
  4. Repeat to get $x_{k+2}$, etc.

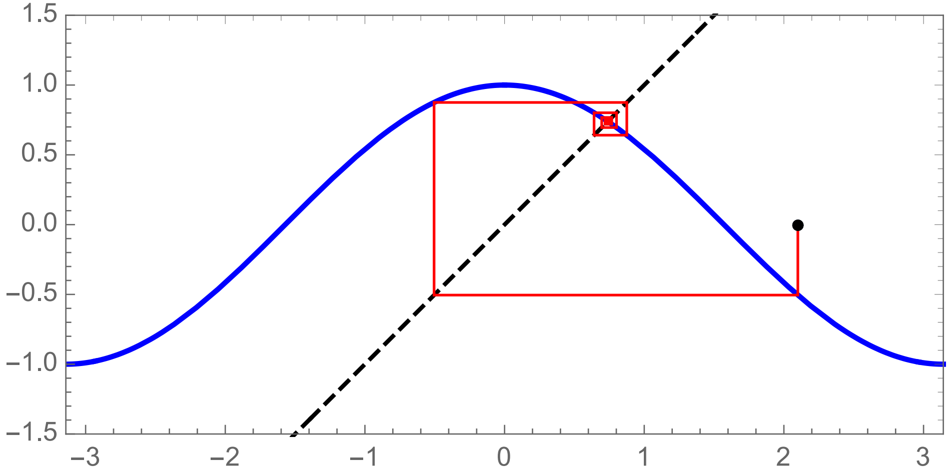


# Example: cosine map

Consider $x_{k+1} = \cos x_k$

Cobweb diagrams for two different starting points:




The iteration converges to a fixed point $x^* = \cos x^*$

$$\approx 0.739$$

# The logistic map

Logistic map, e.g. population dynamics

$$x_{k+1} = r x_k (1 - x_k)$$

- Two fixed points

$$x^* = r x^* (1 - x^*) \implies r x^* \left[ x^* - \left( 1 - \frac{1}{r} \right) \right] = 0 \implies x \in \{0, 1 - \tfrac{1}{r}\}$$

- Jacobian linearization around equilibrium points:

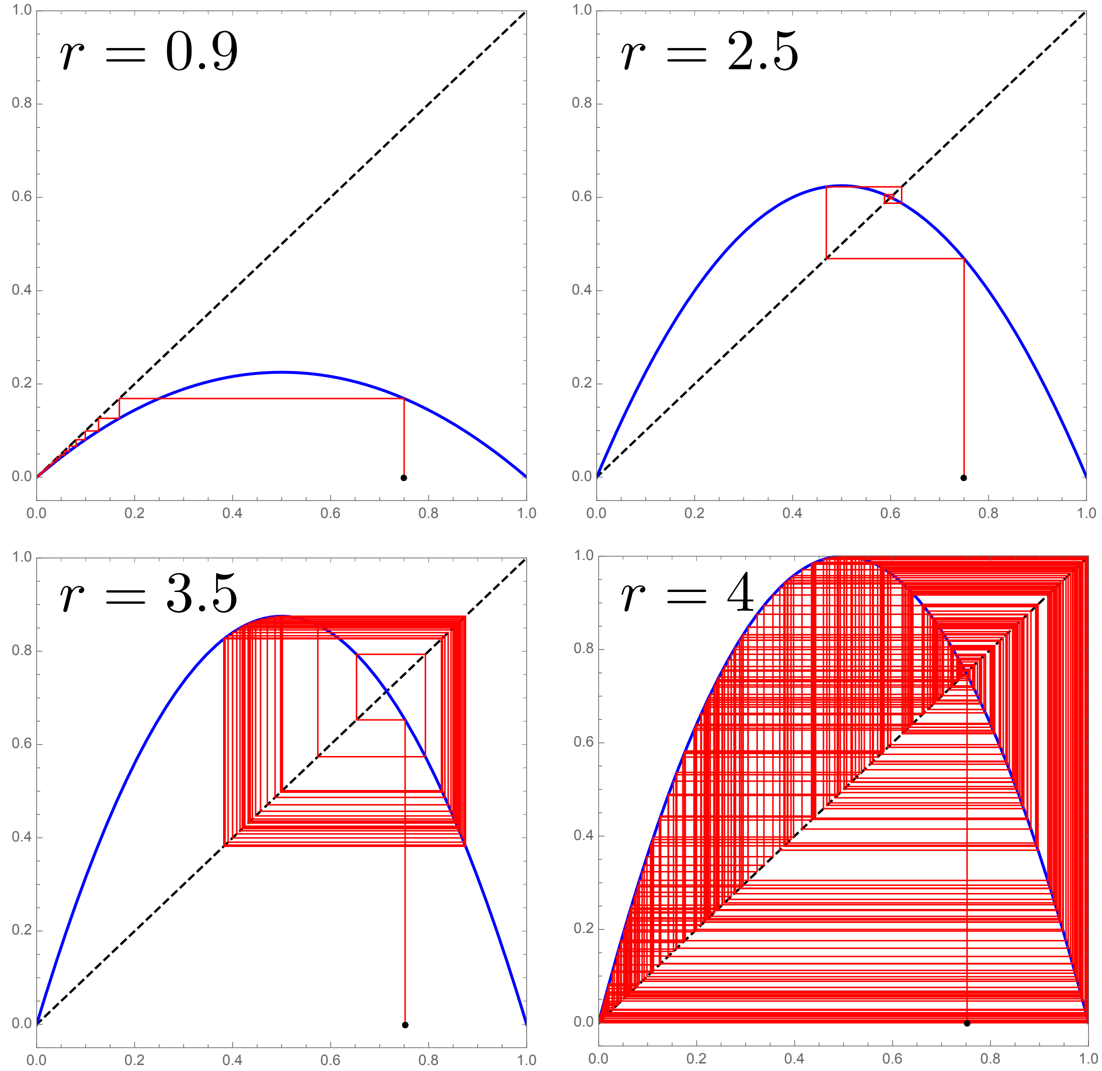
$$w_{k+1} = r w_k (1 - 2x^*) \implies \begin{cases} w_{k+1} = r w_k & \text{at } x^* = 0 \\ w_{k+1} = (2 - r) w_k & \text{at } x^* = 1 - \frac{1}{r} \end{cases}$$

- Behaviour near $x^* = 0$: population dies out if $r < 1$
  population grows if $r > 1$
- What happens at the other equilibrium point?

# Cobwebs for the logistic map

$$x_{k+1} = r x_k (1 - x_k)$$

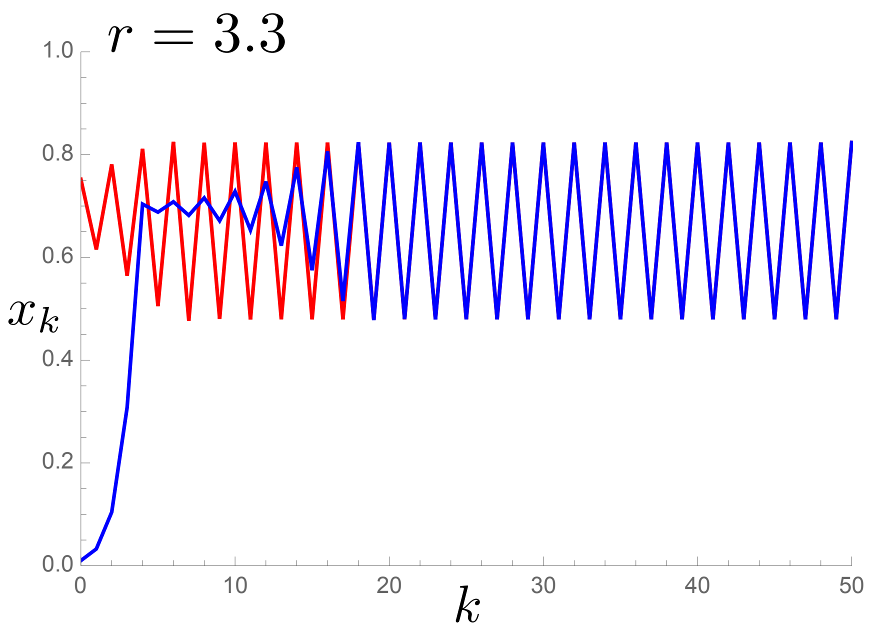
$$x_0 = 0.752$$

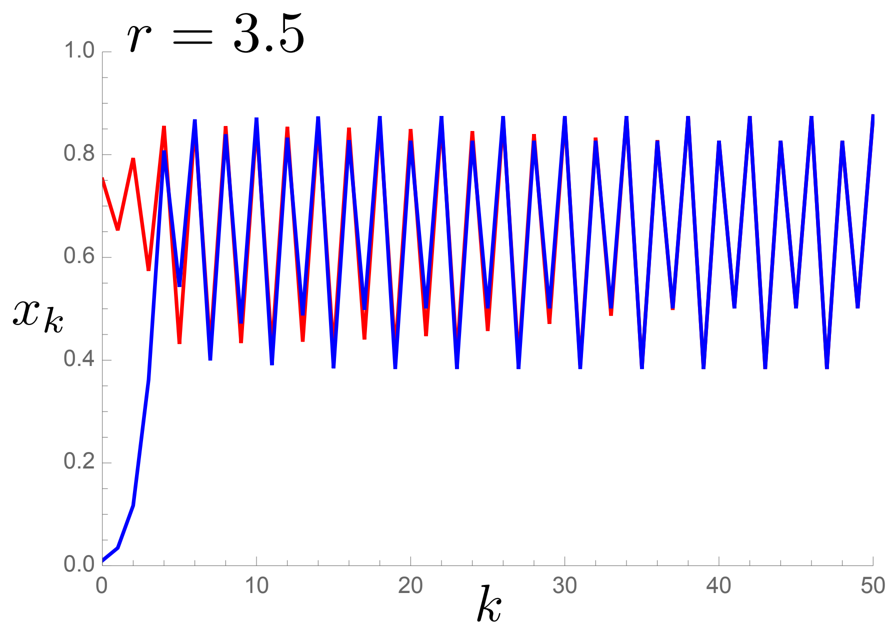


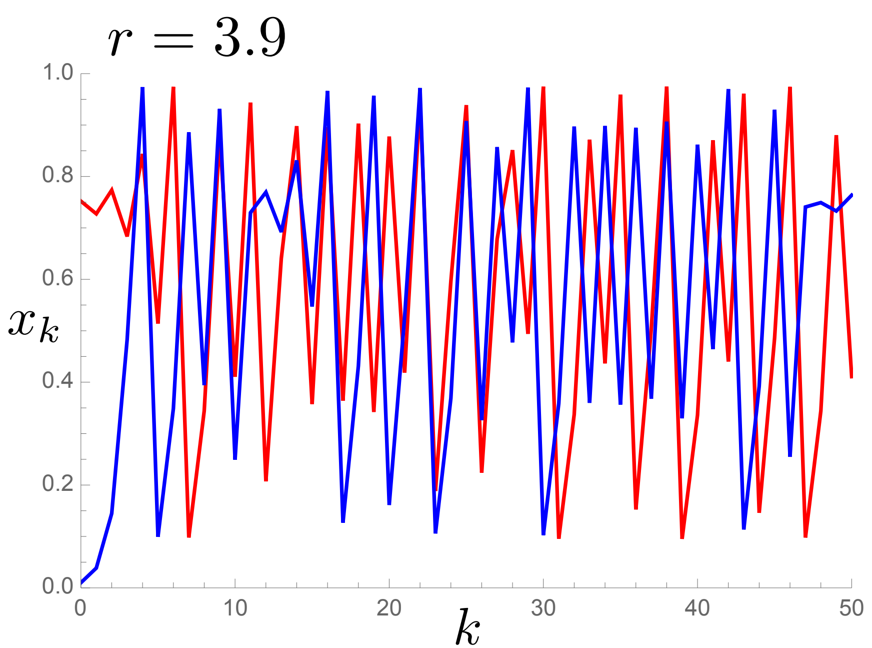

As $r$ increases past 1, fixed point moves away from zero

With further increase in $r$, solution becomes periodic, then aperiodic

# Simulations for the logistic map




- $r = 3.3$: trajectory is periodic
  $r = 3.5$: double-periodic
  $r = 3.9$: aperiodic
- Initial condition $x_0$ doesn't affect results

# Behaviour of the logistic map

Consider points near $x^* = 1 - 1/r$ where the map's linearization is

$$w_{k+1} = (2 - r)w_k$$

- This equilibrium is stable if $|2 - r| \leq 1$, i.e. $1 \leq r \leq 3$
- For $r > 3$, $(2 - r) < -1$, so oscillations appear, with period doubling at fixed values of $r$ as $r$ increases
- The rate of doubling increases until $r$ approaches $\sim 3.57$, at which point the response ceases to be periodic
- As $r$ continues to increase, the aperiodic behaviour continues with brief intermittent 'islands of stability' that are periodic

# Periodicity of the logistic map

- Periodicity can be investigated by considering expressions like

$$x_{k+2} = F\big(F(x_k)\big) = r^2 x_k(1 - x_k)\big[1 - r x_k(1 - x_k)\big]$$

- Fixed points of this map are called 2-cycles (they have period 2) and solve the equation

$$\begin{aligned} x^* &= F\big(F(x^*)\big) = r^2 x^*(1 - x^*)\big[1 - r x^*(1 - x^*)\big] \\ \implies \quad x^* &\in \Big\{0, 1 - \tfrac{1}{r}, \tfrac{1}{2}(1 + \tfrac{1}{r})\big[1 - (\tfrac{r-3}{1+r})^{1/2}\big], \\ &\qquad \tfrac{1}{2}(1 + \tfrac{1}{r})\big[1 + (\tfrac{r-3}{1+r})^{1/2}\big]\Big\} \end{aligned}$$

- A bifurcation occurs at $r = 3$, when the last 2 roots become real; linearizing around these roots $\Rightarrow$ stable for $3 < r \leq 1 + \sqrt{6} \approx 3.45$

- Similar analysis is possible for 4-cycles, 8-cycles, ... but quickly becomes intractable

# Orbit diagrams

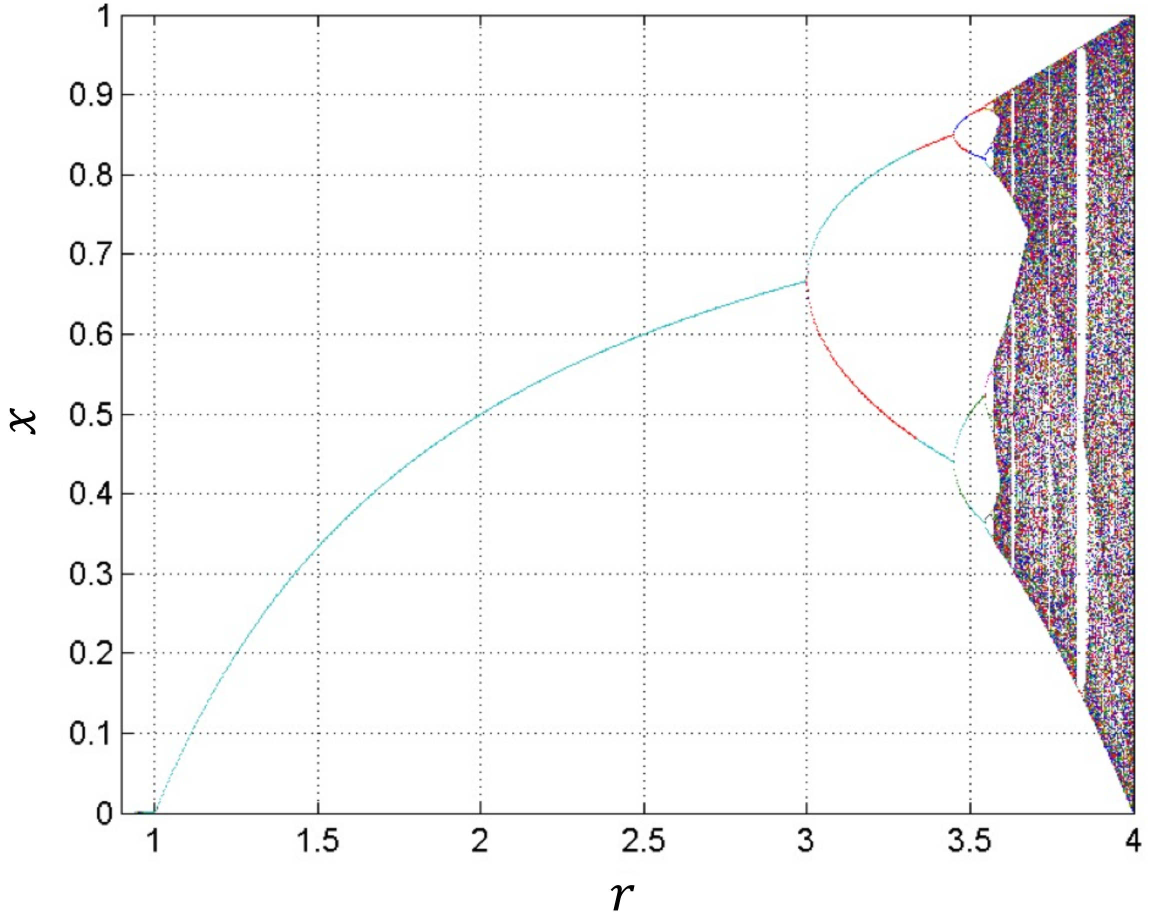
An orbit diagram displays how a map changes with respect to a parameter (and an initial value)

The process

1. Choose a value of $r$ and a starting point $x_0$
2. Iterate the map $m$ times (e.g. $m =$ a few hundred)
3. Record the values obtained when iterating another $n$ times (e.g. $n =$ a few hundred), and plot them on the diagram
4. Change $r$ and repeat

Several plots are given in the lecture notes

# Orbit diagram for the logistic map




Observe the bifurcations and doubling, as well as the descent into aperiodicity

# Chaos

The logistic map illustrates the phenomenon of dynamical **chaos**

Chaos is defined as **aperiodic** long-term behaviour in a **deterministic** system that has a **sensitive dependence on initial conditions**

- Aperiodic long-term behaviour: The trajectories never settle down to fixed points or periodic orbits
- Deterministic system: The trajectory is the solution of an equation with no noise - everything is certain and precise
- Sensitive to initial conditions: Trajectories that pass through points that are close together in phase space diverge with time

# Lyapunov exponent

- The Lyapunov exponent measures how fast solutions that are initially close together in phase space diverge (i.e. it quantifies sensitivity to initial conditions)

- Consider the effect on long-term behaviour when perturbing the initial condition from $x_0$ to $x_0 + w_0$:

$$
\begin{aligned}
x_1 + w_1 &= F(x_0 + w_0) \\
x_2 + w_2 &= F(F(x_0 + w_0)) \\
&\vdots \\
x_k + w_k &= F(\cdots F(x_0 + w_0))
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
w_1 &= F(x_0 + w_0) - F(x_0) \\
w_2 &= F(F(x_0 + w_0)) \\
&\quad - F(F(x_0)) \\
&\vdots \\
w_k &= F(\cdots F(x_0 + w_0)) \\
&\quad - F(\cdots F(x_0))
\end{aligned}
$$

- The Lyapunov exponent $\lambda$ scales the growth rate of $w_k$:

$$|w_k| \approx |w_0| e^{\lambda k}$$

# Determination of Lyapunov exponents

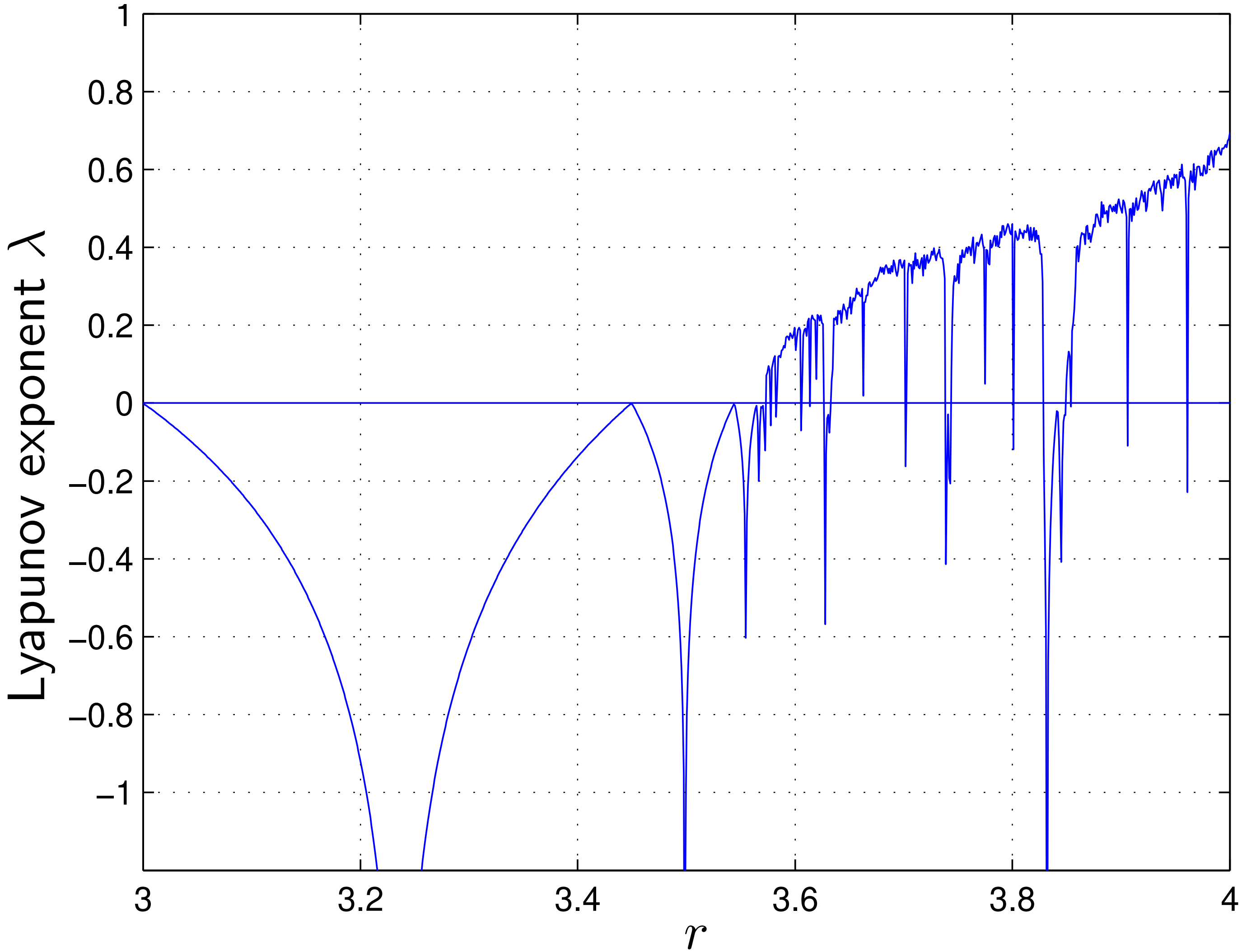
Compute Lyapunov exponents using linearisation:

$$
\begin{aligned}
w_1 &= F(x_0 + w_0) - x_1 \\
&\approx F(x_0) + DF(x_0)w_0 - x_1 = DF(x_0)w_0 \\
w_2 &\approx DF(x_1)w_1 \\
&\approx DF(x_1)DF(x_0)w_0 \\
&\vdots \\
w_k &\approx \prod_{i=1}^{k-1} DF(x_i)w_0
\end{aligned}
$$

The Lyapunov exponent is then given by

$$
\lambda = \lim_{k\to\infty} \lim_{|w_0|\to 0} \frac{1}{k} \ln\left|\frac{w_k}{w_0}\right| \approx \frac{1}{k}\sum_{i=0}^{k-1} \ln|DF(x_i)| \quad \text{(accurate for large } k\text{)}
$$

# Lyapunov exponents for the logistic map




Dips in the exponent indicate periodic solutions

# Chaos in flows

Chaos can also occur in differential systems

The **Lorenz equations**, which model convection in the atmosphere, are a well-known system that exhibits chaos:

$$\begin{aligned} \dot{x} &= \sigma(y - x) & \sigma > 0 &: \text{Prandtl number} \\ \dot{y} &= rx - y - xz & r > 0 &: \text{Rayleigh number} \\ \dot{z} &= xy - bz & b > 0 &: \text{constant parameter} \end{aligned}$$

- If $r < 1$, the only equilibrium is the origin
- If $r > 1$ two more equilibria appear via a pitchfork bifurcation:

$$(x^*, y^*, z^*) \in \Big\{ \big(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\big), \\ \big(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\big) \Big\}$$

# Volume contraction in the Lorenz equations

From the Lorenz equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

we can compute the divergence of the flow:

$$\nabla \cdot \mathbf{f} = \frac{\partial \sigma(y-x)}{\partial x} + \frac{\partial rx - y - xz}{\partial y} + \frac{\partial xy - bz}{\partial z} = -(1 + b + \sigma)$$

- Since $\nabla \cdot \mathbf{f} < 0$, we have, for any control volume $V$:

$$\int_V \nabla \cdot \mathbf{f} \, dV = \oint_{\partial V} \mathbf{f} \cdot d\mathbf{S} = \text{flow out} - \text{flow in} < 0$$

- So trajectories converge to a zero-volume region of phase space but this zero-volume solution is neither a point or a limit cycle!

# Lorenz equations: stability 1

- A solution of the Lorenz equations cannot have unstable equilibrium points or unstable periodic orbits – both of these types of solution imply expansion of the state space, not contraction

- Any fixed points must therefore be stable or saddles
  if there are limit cycles, they must be stable

- Linearization reveals that the origin is a stable node for $r < 1$ and a saddle for $r > 1$

- For $r < 1$ Lyapunov analysis shows that the system is globally asymptotically stable – there are no limit cycles and all trajectories fall into the origin

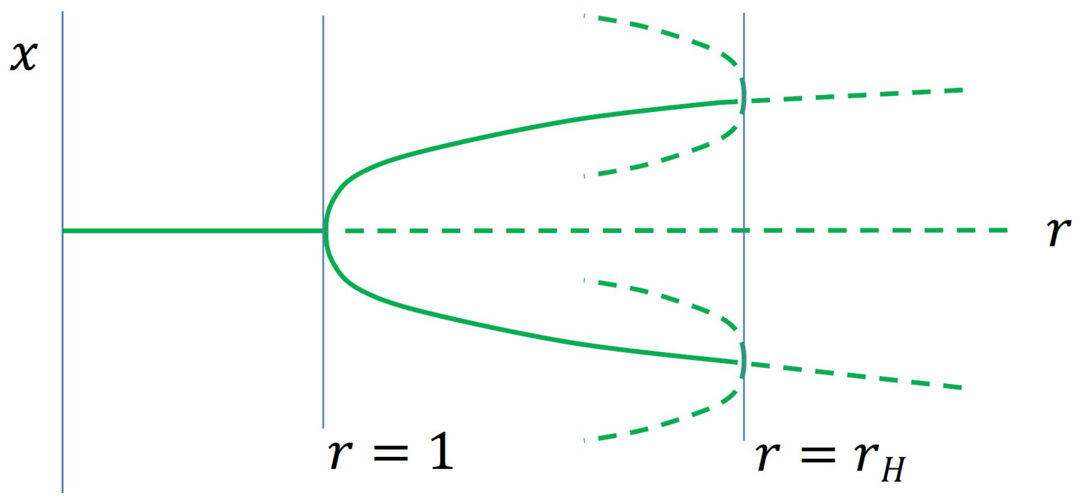
# Lorenz equations: stability 2

- For $r > 1$ two new equilibria appear: assume $\sigma - b > 1$ and let

$$r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

then for

  - $1 < r < r_H$ the two new equilibria are stable
  - $r = r_H$ they undergo a Hopf bifurcation
  - $r > r_H$ there is a saddle point (and no attractors nearby)



# Strange attractor

Here we have a situation where:

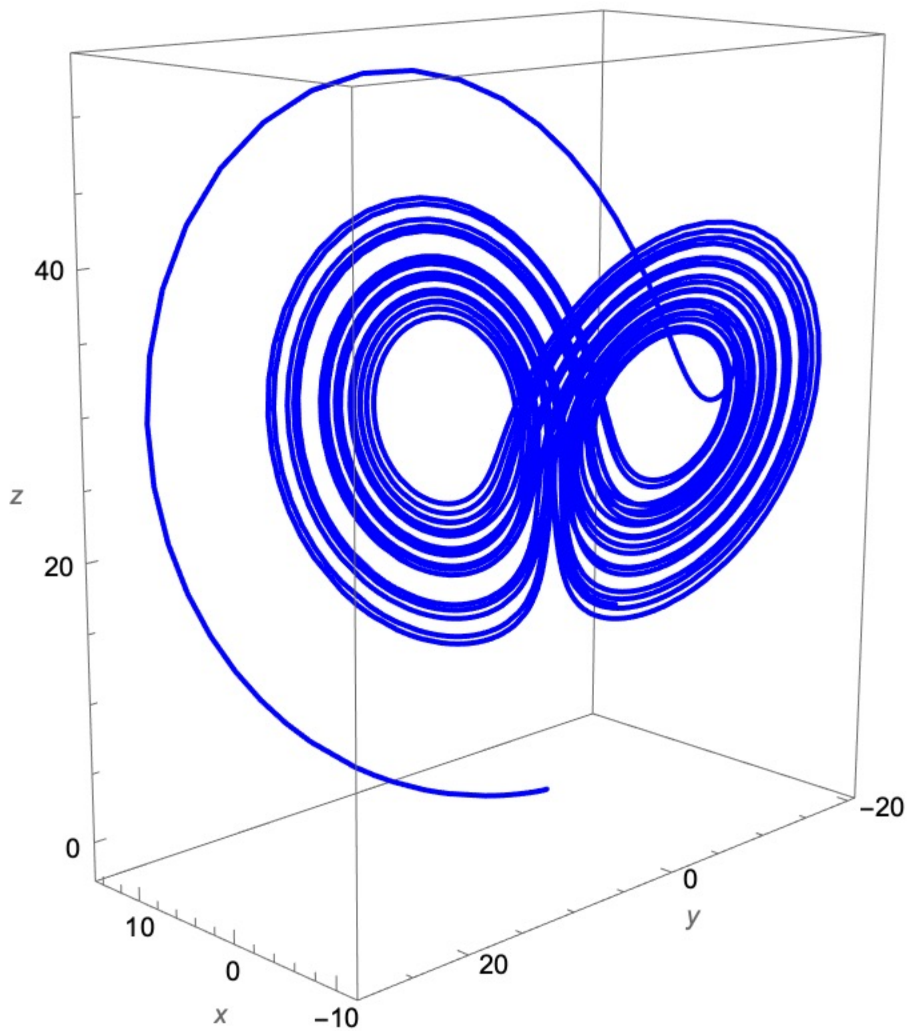
- the volume occupied by neighbouring trajectories in phase space is always contracting
- there are no stable equilibrium points
- there are no stable limit cycles for $r > r_H$ (Lorenz proved this)
- trajectories cannot go to infinity

We conclude that there must be a zero-volume object that attracts the trajectories - we call this a **strange attractor**

# The Lorenz butterfly

Plot of solution trajectory for $\sigma = 3$, $r = 29.4$, $b = 1$
initial condition $(x, y, z) = (0.1, 0.1, 0.1)$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$



# The Mandelbrot set

We now revisit a map defined over the complex numbers by

$$z_{k+1} = z_k^2 + c, \quad c \in \mathbb{C}$$

- The point $c$ belongs to the **Mandelbrot set** if this iteration remains bounded for all $k$ when starting at $z_0 = 0$
- To illustrate the set, we use different colours to show the rate of divergence at $c$
- The Mandelbrot set is a strange attractor for the map; colours indicate how close the points are to the attractor

# Thank you!