In this lecture we consider limit cycles in detail.

We exclude separatrix cycles (in particular homoclinic and heteroclinic connections connecting \( \alpha \) to \( \omega \) points – going from an \( \alpha \) to an \( \omega \) point takes an infinite time – not exactly periodic!)

**Definition:** A solution of \( \dot{x} = f(x) \) through \( x_0 \) is said to be periodic if there exists a \( T > 0 \) such that \( \phi(t, x_0) = \phi(t + T, x_0) \) for all \( t \in \mathbb{R} \).

The minimum such \( T \) is called the period of the periodic orbit.
Proving a periodic orbit does not exist

We may wish to prove that a particular second order dynamical system does not posses periodic orbits.

**Bendixson’s criterion:** Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$.

If \( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \) is **not identically zero** and **does not change sign** within a simply connected region $D$ of the phase plane, then the system has **no closed orbits** in $D$. 
Outline of proof

• Assume a closed orbit $\Gamma \subset D$ exists. Then the orbit is a parametric curve in $t$ such that $\frac{dy}{dx} = \frac{g}{f}$.

• So on a closed orbit $\Gamma$ we get $f \, dy = g \, dx \Rightarrow \oint_{\Gamma} (g \, dx - f \, dy) = 0$

• Using Stoke’s theorem we then have

$$\int_S \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \, dx \, dy = 0$$

where $S \subset D$ is the region enclosed by $\Gamma$

• But if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is never zero, then it must have the same sign all over $D$ and so the integral cannot be zero.
Dulac’s criterion

We consider the same differential equations but now allow both $f$ and $g$ to be multiplied by another function $B$.

Dulac’s criterion: Let $B(x, y)$ be a continuously differentiable function, defined on a region $D \subset \mathbb{R}^2$.

If $\frac{\partial (Bf)}{\partial x} + \frac{\partial (Bg)}{\partial y}$ is not identically zero and does not change sign in $D$, then the system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ has no closed orbits in $D$. 
Example using Bendixson’s criterion

Let

\[ \dot{x} = y \quad \text{def} \quad f(x, y) \]
\[ \dot{y} = x - x^3 - \gamma y \quad \text{def} \quad g(x, y), \quad \gamma \geq 0 \]

Then

\[ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\gamma \]

Thus for \( \gamma \neq 0 \) there are no closed orbits.

For \( \gamma = 0 \), there is an energy function \( \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} \) and the system is Hamiltonian, so we can study the trajectories using level sets (see Lec. 4).
Second example

Let

\[ \dot{x} = y \overset{\text{def}}{=} f(x, y) \]
\[ \dot{y} = x - x^3 - \gamma y + x^2 y \overset{\text{def}}{=} g(x, y), \quad \gamma \geq 0 \]

then

\[ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\gamma + x^2. \]

Linearisation shows that (0,0) is a saddle, and (±1,0) are stable nodes for \( \gamma > 1 \) and unstable nodes for \( 0 \leq \gamma < 1 \).

There can be no closed orbits \textbf{within} regions where \( x \) is very large or very small compared to \( \gamma \), but orbits may pass through these regions – we are thus undecided using Bendixson.
Gradient Systems

• Equations are such that $\dot{x} = -\nabla V$

• Such systems cannot have closed orbits since

$$\dot{V} = \nabla V \cdot \dot{x} = -\nabla V \cdot \nabla V = -|\dot{x}|^2$$

implies that, if $x(t)$ is on a closed orbit of period $T$, then

$$V(x(T)) - V(x(0)) = -\int_0^T |\dot{x}|^2 dt \neq 0$$

But on a closed orbit we must have $V(x(T)) = V(x(0))$ implying $|\dot{x}| = 0$ so $x(0)$ must be a fixed point, not an orbit
Index Theory

For two-dimensional systems the equations
\[ \dot{x}_1 = f(x_1, x_2) \]
\[ \dot{x}_2 = g(x_1, x_2) \]
define a vector field or flow.

These vectors make an angle to the \( x_1 \)-axis:
\[ \phi(x_1, x_2) = \tan^{-1}\left(\frac{g(x_1, x_2)}{f(x_1, x_2)}\right) \]
Index Theory

The index of a (non-intersecting, simple) curve $\Gamma$, $I(\Gamma)$ is defined by

$$I(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} d\varphi(x_1, x_2)$$

Follow the curve $\Gamma$ around anti-clockwise and measure how many times the vectors on the curve rotate anti-clockwise during one rotation.
Properties of Indices

• The index is an integer (you must rotate by a multiple of $2\pi$ to get back to the start).

• If there are no equilibria inside $\Gamma$, then the index is zero ($I(\Gamma) = 0$).

• If $\Gamma$ coincides with a closed orbit, then the index is 1.

• If $\Gamma$ encloses an isolated saddle equilibrium point, then the index is $-1$. If $\Gamma$ encloses any other isolated equilibrium point then the index is 1.

• The index of a curve $\Gamma$ enclosing multiple isolated equilibrium points is the sum of the indices of the individual equilibrium points enclosed.
Some observations

• Given a closed curve that does not enclose any equilibrium points – can this be a trajectory?

**No:** if this curve is a closed orbit then its index is 1, but as there are no equilibrium points the curve has index 0.

• Can there be a closed trajectory surrounding a single saddle node?

**No:** the index of a saddle is $-1$ and that of a closed orbit is 1.
Return to Bendixson example

Earlier problem:

\[
\begin{align*}
\dot{x} &= y \overset{\text{def}}{=} f(x, y) \\
\dot{y} &= x - x^3 - \gamma y + x^2 y \overset{\text{def}}{=} g(x, y), \quad \gamma \geq 0
\end{align*}
\]

Index of curve = 1-1+1=1
this is the only possible closed orbit

Index = 1
Index = -1
Index = 1

\[
x = -\sqrt{\gamma}
\]
\[
x = \sqrt{\gamma}
\]
Matlab simulation $\gamma = 2$

Unstable limit cycle
Another example of reasoning

Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_1 (3 - x_1 - 2x_2) \\
\dot{x}_2 &= x_2 (2 - x_1 - x_2)
\end{align*}
\]

with Jacobian:

\[
\begin{bmatrix}
3 - 2x_1 - 2x_2 & -2x_2 \\
-x_1 & 2 - x_1 - 2x_2
\end{bmatrix}
\]

Equilibrium points:

- \(y_1^* = (0,0)\), Jacobian: \[
\begin{bmatrix}
3 & 0 \\
0 & 2
\end{bmatrix}
\]
  positive eigenvalues (unstable node) so \(I(y_1^*) = 1\)

- \(y_2^* = (0,2)\), Jacobian: \[
\begin{bmatrix}
-1 & 0 \\
-2 & -2
\end{bmatrix}
\]
  negative eigenvalues (stable node) so \(I(y_2^*) = 1\)

- \(y_3^* = (3,0)\), Jacobian: \[
\begin{bmatrix}
-3 & -6 \\
0 & -1
\end{bmatrix}
\]
  negative eigenvalues (stable node) so \(I(y_3^*) = 1\)

- \(y_4^* = (1,1)\), Jacobian: \[
\begin{bmatrix}
-1 & -2 \\
-1 & -1
\end{bmatrix}
\]
  eigenvalues 0.414, -2.414 (saddle) so \(I(y_4^*) = -1\)
Reasoning (continued)

• For closed trajectory we need an index of 1.

• Trajectories cannot cross (curves only meet at equilibrium points) and there are no equilibrium points in the 2\textsuperscript{nd}, 3\textsuperscript{rd}, or 4\textsuperscript{th} quadrants.

• There are trajectories on the $x_1$- and $x_2$-axes, so no trajectory can cross into the 2\textsuperscript{nd}, 3\textsuperscript{rd}, or 4\textsuperscript{th} quadrants. Hence these quadrants are free of any part of a closed trajectory. Thus $(0,0)$, $(0,2)$ and $(3,0)$ cannot lie inside a closed trajectory.

• The point $(1,1)$ is a saddle with index $-1$, so it cannot lie inside a closed trajectory.

• **Conclusion:** there are no closed trajectories.
Graphical illustration of reasoning

Index = 0: not a trajectory

Index = -1: not a trajectory

No trajectory can cross here

No trajectory can cross here
Limit Cycles

- Limit cycles are **isolated** periodic orbits that can be stable or unstable (a cycle around a linear centre is **not** isolated and thus not a limit cycle).

- In the plane a limit cycle is the $\alpha$ or $\omega$ limit set of some trajectory other than itself.

- **Definition**: A periodic orbit $\Gamma$ is said to be **stable** if for every $\epsilon > 0$ there is a neighbourhood $U$ of $\Gamma$ such that for $x \in U$ the distance between $\phi(t, x)$ and $\Gamma$ is less than $\epsilon$. $\Gamma$ is called **asymptotically stable** if it is stable and, for all points $x \in U$, this distance tends to zero as $t$ tends to infinity.
Condition for stability

Let $\dot{x} = f(x)$ have a periodic solution $x = \gamma(t), 0 \leq t \leq T$, then the periodic orbit $\Gamma$ lies on $\gamma(t)$.

- The periodic orbit is asymptotically stable only if
  \[ \int_0^T \nabla f(\gamma(t)) dt \leq 0 \]

- For planar systems, if $\Gamma$ is the $\omega$ limit set of all trajectories in the neighbourhood of $\Gamma$ then it is a **stable** limit cycle.

- If it is the $\alpha$ limit set, then it is an **unstable** limit cycle.

- If it is the $\omega$ limit set for one trajectory and the $\alpha$ limit set for some other trajectory it is called a **semi-stable** limit cycle.
Limit cycle example

Consider
\[
\begin{align*}
\dot{x} &= -y + x(1 - x^2 - y^2)^2 \\
\dot{y} &= x + y(1 - x^2 - y^2)^2
\end{align*}
\]

In polar co-ords:
\[
\begin{align*}
\dot{r} &= r(1 - r^2)^2 \\
\dot{\theta} &= 1
\end{align*}
\]

For \( r \neq 1, \dot{r} > 0 \) and we spiral out.

For \( r = 1, \dot{r} = 0 \) \( \Rightarrow \) a limit cycle, which must be **semi-stable**.
The Poincaré Map

• An extremely important tool for the analysis of dynamical systems, sometimes called the ‘return map’.

• Based on a hyperplane perpendicular to a periodic orbit.

• Consider points close to $x_0$ on the orbit and where those points arrive back in the hyperplane after traversing the orbit. This defines a map $x \mapsto P(x)$

• As the map is iterated, the intersection point $x$ moves in the hyperplane.

• If we are on a periodic orbit then the return point is the original point.
The Poincaré Map

Hyperplane $\Sigma$

$P(x)$

$\Gamma$

$x_0$

$x$
Example

\[ \dot{x} = -y + x(1 - x^2 - y^2) \]
\[ \dot{y} = x + y(1 - x^2 - y^2) \]

In polar co-ords:

\[ \dot{r} = r(1 - r^2) \]
\[ \dot{\theta} = 1 \]

This has stable limit cycle, which is an attractor for \( \mathbb{R}^2 - \{0\} \) (see Lec. 5)

Solving the equations:

\[ \int \frac{dr}{r(1 - r^2)} = \int dt \Rightarrow r = \frac{1}{\sqrt{1 - \left(\frac{1}{r_0^2} - 1\right)e^{-2t}}} \]
\[ \theta = t + \theta_0 \]
Poincaré Map

The hyperplane $\Sigma$ is the ray $\theta = \theta_0$ through the origin that is crossed every $2\pi$ seconds.

Thus

$$P(r_0) = \frac{1}{\sqrt{1 - \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}}}$$

and $P(1) = 1$ (a fixed point).

Also

$$\left.\frac{dP}{dr}\right|_{r=1} = e^{-4\pi} < 1$$

so the limit cycle is stable.